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# Double covers of cubic graphs with oddness 4

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## Abstract

We prove that a cubic 2-connected graph which has a 2-factor containing exactly 4 odd cycles has a cycle double cover.

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## 1. Introduction

For a graph  $G$  we let  $v(G)$  denote the number of vertices in  $G$ . We let  $E_v$  denote the set of edges incident with a vertex  $v$ , and we let  $N_G(v)$  be the set of vertices which are neighbours to  $v$ . For a subset  $X \subseteq V(G)$ , or a subgraph  $X \subseteq G$  we let  $\partial X$  be the set of edges with one end in  $X$  and the other in  $V(G) \setminus X$  and we let  $d_G(X)$  be the number of edges in this set. For  $l \geq 0$  we let  $v_l(G)$  be the number of vertices of degree  $l$ , and we let  $v_{\geq l}(G)$  (resp.,  $v_{\leq l}(G)$ ) be the number of vertices of degree at least  $l$  (resp., at most  $l$ ).

A *bridge* in a graph is an edge whose deletion results in a graph with more components. We say that a cubic graph is *cyclically  $k$ -edge connected* if for any separating subset  $A \subset E(G)$  where  $|A| < k$ , it holds that at most one component of  $G \setminus A$  is not a tree. A *snark* is defined to be a cubic, cyclically 4-edge connected graph  $G$  having girth at least 5 and chromatic index 4; that is,  $\chi'(G) = 4$ . Here the chromatic index of a graph  $G$ , denoted  $\chi'(G)$ , is the smallest number of colours which can be assigned to the edges of  $G$  so that no 2 edges of

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the same colour meet at a vertex. The smallest snark is known to be the Petersen graph  $P_{10}$ , which has 10 vertices. It is also known that there are no snarks with 12, 14, or 16 vertices, but there are 2 snarks with 18 vertices, 6 snarks with 20 vertices, and 20 snarks with 22 vertices (see [1,3]).

We shall refer to a subgraph all of whose degrees are even as *circuit*. On the other hand, a connected, 2-regular subgraph will be called a *cycle*. A collection of cycles (resp., circuits) which covers the edges of a graph exactly twice will be called a *cycle double cover* (resp., *circuit double cover*). A  $k$ -cycle (resp.,  $k$ -circuit) double cover is a cycle (resp., circuit) double cover with at most  $k$  cycles (resp., circuits).

For a cubic bridgeless graph  $G$ , we can partition the vertices by a set of vertices  $X$  (possibly empty) and a set of disjoint cycles  $\mathcal{C}$ . We call the pair  $(X, \mathcal{C})$  a *pseudo 2-factor* of  $G$ . We define the *oddness* of  $G$ , denoted  $o(G)$ , to be the minimum  $k$  such that there is a pseudo 2-factor  $(X, \mathcal{C})$  where  $|X|$  plus the number of odd cycles in  $\mathcal{C}$  equals  $k$ . This definition extends the one given by Huck and Kochol [8] who proved the following:

**Theorem 1.1** (Huck and Kochol [8]). *Let  $G$  be a cubic, bridgeless graph. If  $G$  has a 2-factor with at most 2 odd cycles, then  $G$  has a 5-circuit double cover.*

As a consequence of this theorem, any cubic bridgeless graph having a hamilton path (a path traversing all vertices) has a double cover. This was also shown in [5]. In this paper, we extend Huck and Kochol's result by showing that for graphs with oddness at most 4, there is a cycle double cover.

**Theorem 1.2.** *Let  $G$  be a cubic bridgeless graph. If  $o(G) = 4$ , then  $G$  has a cycle double cover.*

Suppose that  $G$  is a cubic, bridgeless graph and  $(X, \mathcal{C})$  is a pseudo 2-factor of  $G$ . We form the graph  $G_{\mathcal{C}}$  by contracting every cycle of  $\mathcal{C}$  so that they become vertices. We call a bridgeless subgraph  $A_{\mathcal{C}} \subset G_{\mathcal{C}}$  a *degree-compatible subgraph* of  $G_{\mathcal{C}}$  if the odd vertices of  $A_{\mathcal{C}}$  are exactly the odd vertices of  $G_{\mathcal{C}}$ . Given  $v \in V(G_{\mathcal{C}}) \setminus X$ , we let  $C(v) \in \mathcal{C}$  denote the corresponding cycle in  $G$ . Any subgraph of  $G_{\mathcal{C}}$  will be given the subscript  $\mathcal{C}$ , and given a subgraph  $J_{\mathcal{C}} \subseteq G_{\mathcal{C}}$ , we let  $J$  be the subgraph of  $G$  by taking the union of  $C(v)$ ,  $v \in V(J_{\mathcal{C}}) \setminus X$  together with the vertices of  $X$  belonging to  $J_{\mathcal{C}}$  and edges of  $G$  corresponding to edges in  $J_{\mathcal{C}}$ . We let  $h(J)$  denote the graph obtained from  $J$  by suppressing all vertices of degree 2. If  $J_{\mathcal{C}}$  is a subgraph of  $G_{\mathcal{C}}$ , then for  $v \in V(J_{\mathcal{C}}) \setminus X$  we let  $C_{h(J)}(v)$  be the cycle in  $h(J)$  corresponding to  $v$ .

For each subgraph  $J_{\mathcal{C}}$  of  $G_{\mathcal{C}}$  we let  $p_{h(J)} : E(h(J)) \rightarrow \{1, 2\}$  be a weighting for  $h(J)$  where

$$p_{h(J)}(e) = \begin{cases} 1 & \text{if } e \in \bigcup_{v \in V(J_{\mathcal{C}}) \setminus X} C_{h(J)}(v), \\ 2 & \text{otherwise.} \end{cases}$$

If there is a collection of cycles in  $h(J)$  which covers each edge  $e \in E(h(J))$  exactly  $p_{h(J)}(e)$  times, then we say that  $h(J)$  is  $\mathcal{C}$ -compatible.

Huck [7] proved independently the above theorem, showing not only that  $G$  has a double cover, but also showing that it has a 5-circuit double cover. His proof is long and complicated.

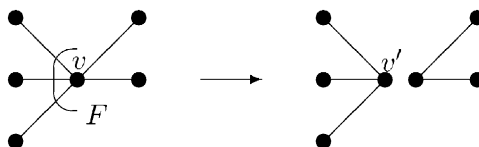


Fig. 1. Splitting the edges of  $F$  away from  $v$ .

This paper presents a more cohesive approach which utilizes splitting and expansion operations to show the following (Theorem 6.1): for a cubic, bridgeless graph  $G$ , if  $o(G) \leq 4$ , then either one can find a degree compatible subgraph  $H_C$  of  $G_C$  such that  $h(H)$  is  $\mathcal{C}$ -compatible, or  $G$  has a non-trivial 3-edge cut.

*Note:* With some extra work, one can show that the theorem stated above is still true even if we replace the condition “ $h(H)$  is  $\mathcal{C}$ -compatible” with  $\chi'(h(H)) = 3$ . Using this, one can strengthen Theorem 1.2 to yield Huck’s result.

The initial steps in the proof of Theorem 1.2 use a “splitting” operation for vertices. Let  $G$  be a graph and suppose  $v \in V(G)$  and  $F \subseteq E_v$ . We define a new graph  $G_{[v;F]}$  by splitting the edges of  $F$  away from  $v$  and creating a new vertex  $v'$  whose incident edges are those of  $F$ .

We call this operation a *splitting* of  $F$  at  $v$  (see Fig. 1). The following theorem (see [4] or [9]) tells us when splitting is possible without creating bridges.

**Theorem 1.3.** *Let  $G$  be a connected bridgeless graph. Suppose  $v \in V(G)$  where  $d_G(v) \geq 4$  and let  $e_0, e_1, e_2 \in E_v$ . Then either  $G_{[v;\{e_0, e_1\}]}$  or  $G_{[v;\{e_0, e_2\}]}$  is connected and bridgeless or  $G_{[v;\{e_1, e_2, e_3\}]}$  has more components than  $G$ .*

The above theorem has the immediate corollary:

**Corollary 1.4.** *Suppose  $G$  is a connected bridgeless graph and  $v \in V(G)$  where  $d_G(v) = 4$  and  $e_0, e_1, e_2 \in E_v$ . Then either  $G_{[v;\{e_0, e_1\}]}$  or  $G_{[v;\{e_0, e_2\}]}$  is connected and bridgeless.*

Let  $G$  be a cubic graph and suppose  $C = v_0e_0v_1e_1v_2e_2v_3e_3v_0$  is a 4-cycle. We create a new cubic graph by deleting the edges  $e_1$  and  $e_3$  and suppressing the resulting vertices of degree 2. Such a graph we denote by  $G \oplus_{\square} \{e_1, e_3\}$ . We call the corresponding operation a  $\square$ -reduction.

**Lemma 1.5.** *Suppose  $G$  is a 2-connected cubic graph and  $C = v_0e_0v_1e_1v_2e_2v_3e_3v_0$  is a 4-cycle. Then either  $G \oplus_{\square} \{e_1, e_3\}$  or  $G \oplus_{\square} \{e_0, e_2\}$  is 2-connected.*

**Proof.** Let  $G$  and  $C$  be as in the statement of the lemma. If  $C$  contains a chord, then the result is clear. We suppose therefore that  $C$  has no chords and we contract the edges of  $C$  so that it becomes a single vertex  $v$  which has degree 4. Let  $G'$  be the resulting graph and suppose  $v$  has incident edges  $f_0, f_1, f_2, f_3$ . Here  $f_i$  corresponds to an edge in  $G$  incident with  $v_i$ . By Corollary 1.4, either  $G_{[v;\{f_0, f_3\}]}$  or  $G_{[v;\{f_0, f_1\}]}$  is connected and bridgeless. This in turn implies that either  $G \oplus_{\square} \{e_0, e_2\}$  or  $G \oplus_{\square} \{e_1, e_3\}$  is 2-connected.  $\square$

The next lemma is a basic observation about  $\square$ -reductions and colourings. The proof is left to the reader.

**Lemma 1.6.** *Suppose  $G$  is a cubic graph and let  $H$  be a cubic graph obtained from  $G$  via a  $\square$ -reduction. Then  $\chi'(G) = 3$  if  $\chi'(H) = 3$ .*

Combining Lemmas 1.5 and 1.6 we obtain:

**Lemma 1.7.** *Suppose  $G$  is a 2-connected cubic graph having disjoint 4-cycles  $C_1, \dots, C_k$ . There exist  $\square$ -reductions on each 4-cycle  $C_1, \dots, C_k$  such that after performing these reductions, we obtain a 2-connected cubic graph  $H$ . Moreover, if  $\chi'(H) = 3$ , then  $\chi'(G) = 3$ .*

**Corollary 1.8.** *Suppose  $G$  is a 2-connected cubic graph having disjoint 4-cycles  $C_1, \dots, C_k$ . If  $G \setminus (C_1 \cup \dots \cup C_k)$  has at most 8 vertices, then  $\chi'(G) = 3$ .*

**Proof.** By Lemma 1.7 there exist  $\square$ -reductions on each 4-cycle  $C_i$ ,  $i = 1, 2, \dots, k$  such that after performing these reductions we obtain a 2-connected cubic graph  $H$ . Since  $G \setminus (C_1 \cup \dots \cup C_k)$  has at most 8 vertices, we have that  $v(H) \leq 8$ . This means that  $\chi'(H) = 3$ , since the smallest 2-connected cubic graph with chromatic index 4 is  $P_{10}$ . Now Lemma 1.7 implies that  $\chi'(G) = 3$ .  $\square$

## 2. Reductions and extensions

Let  $G$  be a 2-connected cubic graph having a 2-edge cut  $\partial X = \{e, f\}$  where  $e = uu'$ ,  $f = vv'$ , and  $u, v \in X$ . We define a new graph by deleting  $e$  and  $f$  and adding new edges  $e' = uv$  and  $f' = u'v'$ , and we denote this graph by  $G \oplus \{e, f\}$ . We call the corresponding operation a 2-edge reduction. If  $G$  has a 3-edge cut  $\partial X = \{e_1, e_2, e_3\}$  where  $e_i = u_i v_i$ ,  $u_i \in X$ ,  $i = 1, 2, 3$ , then we can define a new graph by deleting  $e_i$ ,  $i = 1, 2, 3$  and adding new vertices  $u$  and  $v$  together with edges  $uu_i$  and  $vv_i$ ,  $i = 1, 2, 3$ . We denote this graph by  $G \oplus \{e_1, e_2, e_3\}$ . We call the corresponding operation a 3-edge reduction.

Suppose  $G$  has a triangle  $T = v_1 e_1 v_2 e_2 v_3 e_3 v_1$ . We define a new cubic graph by contracting (ie. identifying)  $T$  with a single vertex. Such a graph we denote by  $G \oplus_{\Delta}(T)$ . We call the corresponding operation a  $\Delta$ -reduction.

Let  $u_1$  and  $u_2$  be the endvertices of a digon  $D$  in  $G$ . By *digon* we mean a pair of edges inducing a 2-cycle. Let  $N_G(D) = \{u'_1, u'_2\}$  (here we allow for  $u'_1 = u'_2$ ). We define a new graph  $G \oplus_{\circ}(D) = (G \setminus D) \cup \{u'_1 u'_2\}$ . Such an operation we call a  $\circ$ -reduction.

Suppose  $v$  is a vertex of degree 2 which is not incident with a loop. Given  $N_G(v) = \{v_1, v_2\}$  (with possibly  $v_1 = v_2$ ) we define a new graph  $G \oplus_{\vee}(v) = (G \setminus v) \cup \{v_1 v_2\}$ . We call the corresponding operation a  $\vee$ -reduction.

Suppose  $G$  and  $H$  are graphs. We say that  $G$  and  $H$  are *homeomorphic* if one graph can be obtained from the other via  $\vee$ -reductions and subdividing edges.

If  $G$  is a graph having no components which are cycles, then we can perform successive  $\vee$ -reductions on  $G$  to obtain a graph with no vertices of degree 2. This graph is seen to be

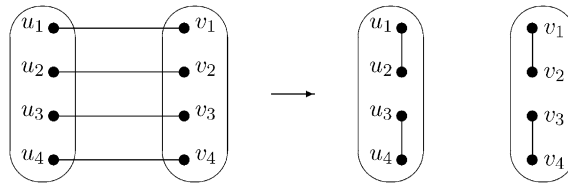


Fig. 2. 4-edge reduction.

the unique graph homeomorphic to  $G$  which has no vertices of degree 2. We denote such a graph by  $h(G)$ . We define the *homeomorph chromatic index* of  $G$ , denoted  $\chi'_h(G)$ , by  $\chi'_h(G) = \chi'(h(G))$ .

Let  $G$  be a cubic graph, and let  $e \in E(G)$  be such that  $e$  is not incident with any loops. We define a graph  $G \oplus e = h(G \setminus e)$ . We obtain  $G \oplus e$  from  $G$  via an operation which we refer to as an *edge-reduction*. The following is a standard result and we refer the reader to [6], chapter 3.

**Theorem 2.1.** *Suppose  $G$  is a bridgeless cubic graph.*

- (a) *If  $G'$  is obtained from  $G$  via a 2- or 3-edge reduction, or via a  $\nabla$ -,  $\circ$ -, or  $\Delta$ -reduction, then  $\chi'(G') = \chi'(G)$ .*
- (b) *If  $G'$  is obtained from  $G$  via a  $\square$ -reduction, then  $\chi'(G') = 4$  if  $\chi'(G) = 4$ .*

**Corollary 2.2.** *Let  $G$  be a cubic graph and let  $G'$  be a cubic graph obtained from  $G$  via a sequence of 2-, 3-,  $\nabla$ -,  $\circ$ -, or  $\Delta$ -reductions. If  $\chi'(G') = 3$ , then  $\chi'(G) = 3$ .*

Suppose we are given a 4-edge cut  $\partial X$  where we order the edges as  $e_1, e_2, e_3, e_4$ , and  $e_i = u_i v_i$ ,  $u_i \in X$ ,  $i = 1, 2, 3, 4$ . We define a new graph, denoted  $G \oplus (e_1, e_2, e_3, e_4)$ , where we delete the edges  $e_i$ ,  $i = 1, 2, 3, 4$  and add the edges  $u_1 u_2, u_3 u_4, v_1 v_2, v_3 v_4$ . We call the corresponding operation a *4-edge reduction* (see Fig. 2).

Similarly, given a 5-edge cut  $\partial X$ , if we order the edges of  $\partial X$  as  $e_1, e_2, e_3, e_4, e_5$  where  $e_i = u_i v_i$ ,  $u_i \in X$ ,  $i = 1, \dots, 5$ , we define a new graph, denoted  $G \oplus (e_1, e_2, e_3, e_4, e_5)$  by first deleting  $e_1, \dots, e_5$ , and then adding edges  $u_4 u_5, v_4 v_5$ , and 2 new vertices  $u$  and  $v$  together with the edges  $u_i u, v_i v$ ,  $i = 1, 2, 3$ . The corresponding operation we call a *5-edge reduction* (see Fig. 3).

We define an insertion operation in the following way: we subdivide an edge of a graph  $G$  inserting a vertex  $u$ , and then subdivide a new edge in the resulting graph, inserting another vertex  $v$ . We then add an edge  $e = uv$ . The combined operation is called an *edge-insertion* operation, which we denote by  $G \odot e$ . If we insert edges  $e_1, \dots, e_k$  successively in  $G$ , then we denote the resulting graph by  $G \odot (e_1, \dots, e_k)$ , or in the case where  $S$  is a subset of edges to be inserted, we let  $G \odot S$  denote the resulting graph.

We define a corresponding insertion operation for vertices, whereby we subdivide edges 3 times in succession, inserting vertices  $u_1, u_2$ , and  $u_3$ . We then add a vertex  $v$  and join it to  $u_1, u_2$ , and  $u_3$  by edges. The operation is called a *vertex-insertion* operation, and we denote the resulting graph by  $G \odot v$ .

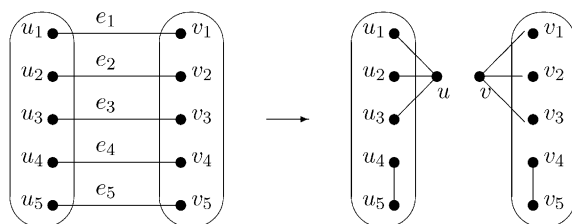
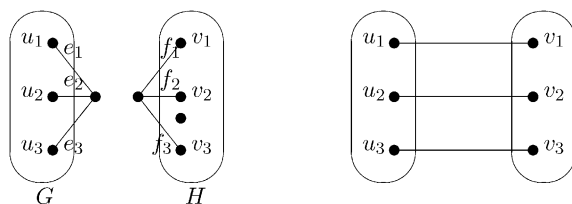


Fig. 3. 5-edge reduction.

Fig. 4. Vertex expansion at  $u$ .

For each of the reduction operations defined above, we can define the reverse operation, namely, an *expansion* operation. Suppose  $G$  is a cubic graph and let  $e = u_1u_2 \in E(G)$ . Let  $H$  be a cubic graph and let  $f = v_1v_2 \in E(H)$ . Given that the endvertices of  $e$  and  $f$  are ordered as  $u_1, u_2$  and  $v_1, v_2$ , respectively, we define  $(G; u_1, u_2; e) \otimes (H; v_1, v_2; f) = (G \setminus \{e\}) \cup (H \setminus \{f\}) \cup \{u_1v_1, u_2v_2\}$  and the corresponding operation we call a *2-edge expansion*.

Suppose  $u \in V(G)$ . Let  $e_1, e_2, e_3$  be an ordering of the edges incident to  $u$  where  $e_i = u_iu$ ,  $i = 1, 2, 3$ . Let  $H$  be a cubic graph and let  $v \in V(H)$ . We suppose  $f_1, f_2, f_3$  is an ordering of the edges incident to  $v$  where  $f_i = v_iv$ ,  $i = 1, 2, 3$ . We define an operation called a *vertex expansion* at  $u$  whereby we delete  $u$ , and add the graph  $H \setminus v$  together with the edges  $u_iv_i$ ,  $i = 1, 2, 3$ . (see Fig. 4). We denote the resulting graph by  $(G; u; e_1, e_2, e_3) \otimes (H; v; f_1, f_2, f_3)$  and denote the corresponding operation by  $u \rightarrow H$ . Note that a vertex expansion may yield the same graph, for example when  $H$  is a multiple 3-edge. In the case where  $H$  is 3-edge colourable, we refer to the vertex expansion as being *3-chromatic*.

If we perform an expansion at each vertex, then we say that the resulting graph is an *expansion* of  $G$ . If in addition each vertex expansion is 3-chromatic, then we say that the expansion is 3-chromatic.

We may define the reverse operations to 4- and 5-edge reductions as well. Let  $G$  and  $H$  be cubic graphs. Pick 2 non-incident edges of  $G$  which we order as  $e_1, e_2$  where we assume  $e_1 = u_1u_2$ , and  $e_2 = u_3u_4$ . Here we order the endvertices of  $e_1$  and  $e_2$  as  $u_1, u_2$  and  $u_3, u_4$ , respectively. Similarly, we pick 2 edges  $f_1, f_2$  in  $H$  where  $f_1 = v_1v_2$  and  $f_2 = v_3v_4$ . We order the endvertices of  $f_1$  and  $f_2$  as  $v_1, v_2$  and  $v_3, v_4$ , respectively. We define a *4-edge expansion* whereby we delete  $e_1, e_2, f_1, f_2$  from  $G \cup H$  and then add the edges  $u_iv_i$ ,  $i = 1, 2, 3, 4$ . (see Fig. 5) We denote this graph by  $(G; u_1, u_3; e_1, e_2) \otimes (H; v_1, v_3; f_1, f_2)$ .

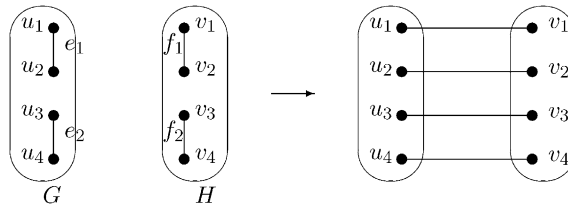


Fig. 5. 4-edge expansion.

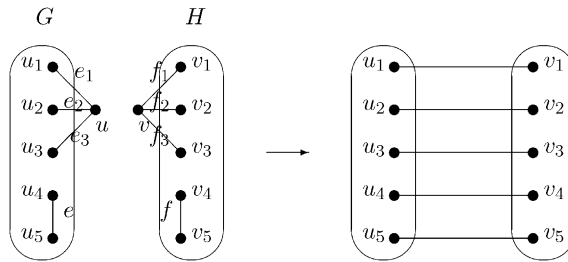


Fig. 6. 5-edge expansion.

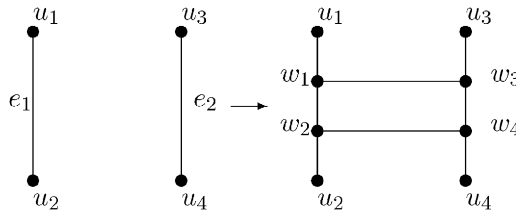


Fig. 7. Expanding a square.

Suppose we are given cubic graphs  $G$  and  $H$  as before. We let  $u$  be a vertex of  $G$  and let  $e \in E(G)$  be an edge non-incident with  $u$ . We order the edges of  $E_u$  as  $e_1, e_2, e_3$  where  $e_i = uu_i$ ,  $i = 1, 2, 3$ . We let  $e = u_4u_5$  where the vertices are ordered as  $u_4, u_5$ . In a similar way, let  $v \in V(H)$  and let  $f \in E(H)$ ,  $f \notin E_v$ . We suppose  $f_1, f_2, f_3$  is an ordering of the edges at  $v$  where  $f_i = vv_i$ ,  $i = 1, 2, 3$ . We let  $f = v_4v_5$  where the endvertices are ordered as  $v_4, v_5$ . We define a 5-edge expansion by deleting  $u, v, e, f$  from  $G \cup H$  and adding the edges  $uv_i$ ,  $i = 1, \dots, 5$  (see Fig. 6). We denote the resulting graph by  $(G; u, u_4; e_1, e_2, e_3, e) \otimes (H; v, v_4; f_1, f_2, f_3, f)$ .

We can define a  $\square$ -expansion in the following way: let  $G$  be a cubic graph and let  $e_1 = u_1u_2 \in E(G)$  and  $e_2 = u_3u_4 \in E(G)$  where the endvertices of  $e_1$  and  $e_2$  are ordered as  $u_1, u_2$  and  $u_3, u_4$ , respectively. We subdivide  $e_1$  by 2 vertices  $w_1$  and  $w_2$ , so that the vertices lie in order  $u_1, w_1, w_2, u_2$ . Next, we subdivided  $e_2$  by the vertices  $w_3$  and  $w_4$  so that the vertices lie in order  $u_3, w_3, w_4, u_4$ . We then add the edges  $w_1w_3$  and  $w_2w_4$ . It is permissible that  $e_1 = e_2$  but  $w_1w_2$  and  $w_3w_4$  must be edges in the subdivided graph. We denote the resulting graph by  $G \otimes_{\square} (e_1, e_2; u_1, u_2; u_3, u_4)$ . See Fig. 7.

Some simple observations are given in the proposition below whose proof is straightforward and left as an exercise for the reader.

**Proposition 2.3.** *Let  $G$  and  $H$  be cubic graphs.*

- (a) *If  $u \in V(G)$  and  $G'$  is the result of a vertex expansion  $u \rightarrow H$ , then  $\chi'(G') = \max\{\chi'(G), \chi'(H)\}$ .*
- (b) *Suppose  $G' = (G; u_1, u_3; e_1, e_2) \otimes (H; v_1, v_3; f_1, f_2)$ . If  $\chi'(G) = \chi'(H) = 3$ , and there exist proper 3-edge colourings  $c_1, c_2 : E(G) \rightarrow \{1, 2, 3\}$  such that  $c_1(e_1) = c_1(e_2)$  and  $c_2(e_1) \neq c_2(e_2)$ , then  $\chi'(G') = 3$ .*
- (c) *Suppose  $G' = (G; u, u_4; e_1, e_2, e_3, e) \otimes (H; v, v_4; f_1, f_2, f_3, f)$ . If  $\chi'(G) = \chi'(H) = 3$  and there exist proper 3-edge colourings  $c_1, c_2, c_3 : E(G) \rightarrow \{1, 2, 3\}$  such that  $c_i(e) = c_i(e_i)$ ,  $i = 1, 2, 3$ , then  $\chi'(G') = 3$ .*
- (d) *Suppose  $G' = G \otimes_{\square} (e_1, e_2; u_1, u_2; u_3, u_4)$ . If  $\chi'(G) = 3$ , then  $\chi'(G') = 3$ .*

**Example 2.4.** Suppose  $e = uu' \in E(P_{10})$  where  $N(u) = \{u', u_1, u_2\}$  and  $N(u') = \{u, u_3, u_4\}$ . Let  $P_8 = P_{10} \oplus e$ , and let  $f_1 = u_1u_2 \in E(P_8)$  and  $f_2 = u_3u_4 \in E(P_8)$ . We have that  $\chi'(P_8) = 3$  and moreover,  $P_8$  has 2 proper 3-edge colourings  $c_1, c_2 : E(P_8) \rightarrow \{1, 2, 3\}$  where  $c_1(f_1) = c_1(f_2)$  and  $c_2(f_1) \neq c_2(f_2)$ .

Suppose now that  $v_1a_1v_2a_2v_3$  is a path of length 2 in  $P_{10}$ . Let  $E_{v_1} = \{a_1, b_1, b_2\}$ ,  $E_{v_2} = \{a_1, a_2, b_3\}$ , and  $E_{v_3} = \{a_2, b_4, b_5\}$  where  $b_1 = u_1v_1$ ,  $b_2 = u_2v_1$ ,  $b_3 = u_3v_2$ ,  $b_4 = u_4v_3$ ,  $b_5 = u_5v_3$ . The graph  $G' = P_{10} \otimes \{b_1, b_2, b_3, b_4, b_5\}$  has 2 components  $G'_1$  and  $G'_2$ , where  $G'_1$  is  $P_8$ . The graph  $G'_1$  is obtained from  $P_{10}$  by deleting the vertices  $v_1, v_2, v_3$  and adding a vertex  $u$  together with the edges  $uu_1, uu_2, uu_3$  and  $u_4u_5$ . There exist proper 3-edge colourings  $c_1, c_2, c_3 : E(G) \rightarrow \{1, 2, 3\}$  such that  $c_i(uu_i) = c_i(u_4u_5)$ ,  $i = 1, 2, 3$  (see Fig. 8).

A sequence of  $\square$ -expansions is said to be *disjoint* if each expansion preserves the 4-cycles created in the previous  $\square$ -expansions. Given that we perform any number of disjoint  $\square$ -expansions on  $P_{10}$  the resulting graph is either 3-edge colourable or is a 3-chromatic expansion of  $P_{10}$ . We have something slightly more general:

**Theorem 2.5.** *Let  $Q = P_{10} \odot (e_1, \dots, e_k)$ . Then either  $\chi'(P_{10} \odot S) = 3$  for some ordered subset  $S \subseteq \{e_1, \dots, e_k\}$  or  $Q$  is an expansion of  $P_{10}$ .*

The above theorem follows from results in Section 7. It implies the following result:

**Theorem 2.6.** *Let  $P'_{10}$  be a 3-chromatic expansion of  $P_{10}$  where  $v(P'_{10}) \leq 16$ . Let  $Q$  be a cubic graph obtained from  $P'_{10}$  via a sequence of disjoint  $\square$ -expansions. Then either  $\chi'(Q) = 3$  or  $Q$  is a 3-chromatic expansion of  $P_{10}$ .*

**Proof.** To minimize the burden of details, we shall only prove the case where  $P'_{10} = P_{10}$ , the proof for the general scenario being the same in spirit. The graph  $Q$  is also obtained by inserting edges into  $P_{10}$ . Now by Theorem 2.5, we have that either we obtain a graph  $Q'$  with  $\chi'(Q') = 3$  via a subsequence of edge insertions (in which case  $\chi'(Q) = 3$ ), or  $Q$  is



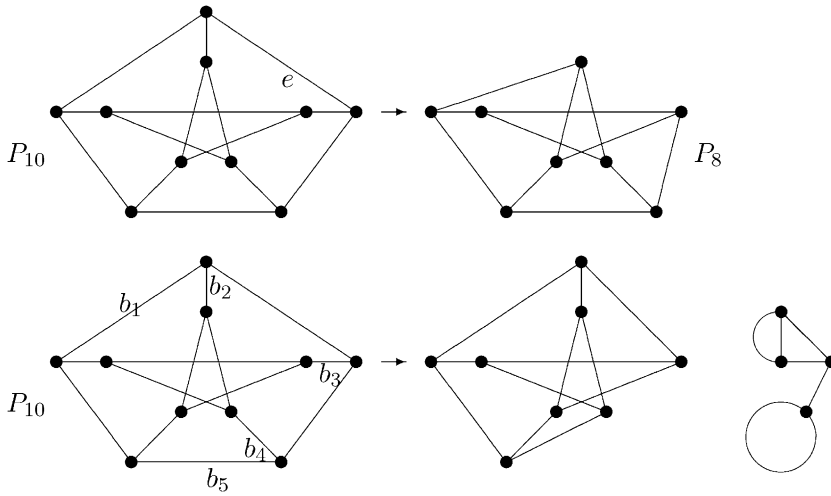


Fig. 8. Edge-reduction on  $P_{10}$  and 5-edge reduction on  $P_{10}$ .

an expansion  $v \rightarrow A_v$ ,  $v \in V(P_{10})$  of  $P_{10}$ . In the former case, we could obtain a 3-edge colourable graph via a subsequence of  $\square$ -expansions, which would imply  $\chi'(Q) = 3$ . In the latter case, each  $A_v$  would be obtained by performing disjoint  $\square$ -expansions on a multiple 3-edge, and thus  $\chi'(A_v) = 3$ . This shows that such an expansion would be 3-chromatic. This completes the proof.  $\square$

Given that  $P_{10}$  is the only snark with 16 or fewer vertices, if  $G$  is a graph with 18 vertices which is not a snark, then either  $\chi'(G) = 3$  or  $G$  is a 3-chromatic expansion of  $P_{10}$ .

**Proposition 2.7.** *Let  $G$  be a 2-connected cubic graph with  $v(G) \leq 16$ . Then either  $\chi'(G) = 3$  or  $G$  is a 3-chromatic expansion of  $P_{10}$ . Moreover, if  $v(G) = 18$ , and  $G$  is not a snark, then the above conclusion is still valid.*

Let  $H_1$  be a cubic graph and let  $u \in V(H_1)$ . Let  $e_1, e_2, e_3$  be an ordering of the edges incident to  $u$  where  $e_i = u_i u$ ,  $i = 1, 2, 3$ . Let  $H_2$  be a cubic graph and let  $v \in V(H_2)$ . We suppose  $f_1, f_2, f_3$  is an ordering of the edges incident to  $v$  where  $f_i = v_i v$ ,  $i = 1, 2, 3$ . We suppose  $\mathcal{C}^1$  and  $\mathcal{C}^2$  are collections of cycles in  $H_1$  and  $H_2$ , respectively, where each  $e_i$  (resp.,  $f_i$ ) is covered twice by cycles in  $\mathcal{C}^1$  (resp.,  $\mathcal{C}^2$ ). We define a splicing operation where the cycles of  $\mathcal{C}^1$  and  $\mathcal{C}^2$  are “spliced” together to form a collection of cycles  $\mathcal{C}$  of  $H = (H_1; u; e_1, e_2, e_3) \otimes (H_2; v; f_1, f_2, f_3)$ . Let  $C_1^1, C_2^1, C_3^1$  be the cycles of  $\mathcal{C}^1$  which contain the pairs of edges  $\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}$ , respectively, and let  $C_1^2, C_2^2, C_3^2$  be the cycles of  $\mathcal{C}^2$  which contain the pairs of edges  $\{f_1, f_2\}, \{f_1, f_3\}, \{f_2, f_3\}$ , respectively. Let  $h_i$  be the edge  $u_i v_i \in E(H)$ ,  $i = 1, 2, 3$ , and let

$$C_1 = (C_1^1 \setminus u) \cup (C_1^2 \setminus v) \cup \{h_1, h_2\},$$

$$C_2 = (C_2^1 \setminus u) \cup (C_2^2 \setminus v) \cup \{h_1, h_3\},$$

$$C_3 = (C_3^1 \setminus u) \cup (C_3^2 \setminus v) \cup \{h_2, h_3\}.$$

Let  $\mathcal{C} = (\mathcal{C}^1 \setminus \{C_1^1, C_2^1, C_3^1\}) \cup (\mathcal{C}^2 \setminus \{C_1^2, C_2^2, C_3^2\}) \cup \{C_1, C_2, C_3\}$ . We call  $\mathcal{C}$  a collection of cycles obtained by *splicing* together  $\mathcal{C}^1$  and  $\mathcal{C}^2$ .

### 3. 3-colourable subgraphs

A circuit which is a vertex-disjoint collection of cycles which partitions the vertices of the graph is called a 2-factor. It is well-known that every bridgeless cubic graph contains a perfect matching and hence also a 2-factor (see [2, p. 79]).

Suppose  $G$  is a 2-connected, cubic, 3-edge colourable graph, and let  $C$  be a circuit of  $G$ . Given  $G$  has a 3-edge colouring with colours 1, 2, 3, we let  $C_{ij}$  be the 2-factor induced by the edges having colours  $i$  or  $j$  where  $i, j = 1, 2, 3$ . Let  $C'_{ij} = C_{ij} \nabla C$ ,  $i, j = 1, 2, 3$ , where ' $\nabla$ ' denotes symmetric difference. Now  $C'_{ij}$ ,  $i, j = 1, 2, 3$  are 3 circuits which cover all the edges of  $G$  twice, except for the edges of  $C$  which are covered once. To summarize:

**Lemma 3.1.** *Let  $G$  be a cubic 3-edge colourable graph and let  $C$  be a circuit of  $G$ . Then there are 3 circuits which cover the edges of  $C$  once, and the edges of  $E(G) \setminus E(C)$  twice.*

We also have a specific variation of this lemma which we will need:

**Lemma 3.2.** *Let  $P'_{10}$  be a 3-chromatic expansion of  $P_{10}$  given by  $v \rightarrow A_v$ ,  $v \in V(P_{10})$ . Let  $\mathcal{C}'$  be a disjoint collection of cycles of  $P'_{10}$  where, with the exception of possibly one cycle, each cycle of  $\mathcal{C}'$  is contained in some  $A_v$ . Then  $P'_{10}$  contains a collection of cycles  $\mathcal{D}'$  which cover the edges of  $\bigcup_{C' \in \mathcal{C}'} E(C')$  once and the other edges of  $P'_{10}$  twice.*

**Proof.** For each  $v \in V(P_{10})$  let  $A'_v$  be the subgraph of  $P'_{10}$  induced by the edges in  $P_{10}$  corresponding to those in  $A_v$ . We shall assume that  $\mathcal{C}'$  contains one cycle  $K'$  which is not contained in any  $A_v$ ,  $v \in V(P_{10})$ . In the case where no such cycle exists, the proof is similar. We first observe that given any cycle  $C$  in  $P_{10}$ , there is a collection of cycles in  $P_{10}$  covering  $C$  once, and the other edges of  $P_{10}$  twice. Let  $K$  be the cycle of  $P_{10}$  corresponding to the cycle  $K'$ . Let  $\mathcal{D}$  be a collection of cycles of  $P_{10}$  which cover  $K$  once and the other edges of  $P_{10}$  twice. For any cycle  $C' \in \mathcal{C}'$ , if  $C'$  intersects  $A'_v$ , then the intersection corresponds to a cycle in  $A_v$ . Moreover, the intersection of the cycles of  $\mathcal{C}'$  with  $A'_v$  corresponds to a disjoint collection of cycles in  $A_v$  which we denote by  $\mathcal{C}_v$ . Since  $A_v$  is 3-edge colourable, Lemma 3.1 implies that there is a collection of cycles  $\mathcal{D}_v$  in  $A_v$  covering the cycles of  $\mathcal{C}_v$  once and the other edges of  $A_v$  twice. One can now splice together the collections  $\mathcal{D}_v$ ,  $v \in V(P_{10})$  with  $\mathcal{D}$  to obtain the desired collection of cycles  $\mathcal{D}'$  of  $P'_{10}$ .  $\square$

**Example 3.3.** Let  $G$  be the cubic graph consisting of  $t$  independent vertices joined to a cycle  $C$  of length  $3t$ . If  $v(G) \leq 16$  (that is,  $t \leq 4$ ), then according to Proposition 2.7 we have that either  $\chi'(G) = 3$  or  $G$  is a 3-chromatic expansion of  $P_{10}$ . It follows by Lemmas

3.1 and 3.2 that there is a collection of cycles in  $G$  covering  $C$  once and the other edges twice.

Suppose that  $G$  is a cubic graph with a pseudo 2-factor  $(X, C)$  and suppose that there are two bridgeless subgraphs  $H_1$  and  $H_2$  where  $G = H_1 \cup H_2$ ,  $E(H_1) \cap E(H_2) = \bigcup_{C \in \mathcal{C}} E(C)$ , and each  $H_i$   $i = 1, 2$  has a collection of cycles  $\mathcal{D}_i$  which cover all the edges of  $H_i$  twice except the edges of  $C$  which are covered once. The collection  $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$  is a cycle double cover of  $G$ . Our strategy for the proof of the main theorem is, when possible, to find two such subgraphs  $H_1$  and  $H_2$ . We note that if  $\chi'_h(H_1) = \chi'_h(H_2) = 3$ , then Lemma 3.1 implies that the desired cycle collections  $\mathcal{D}_1$  and  $\mathcal{D}_2$  exist.

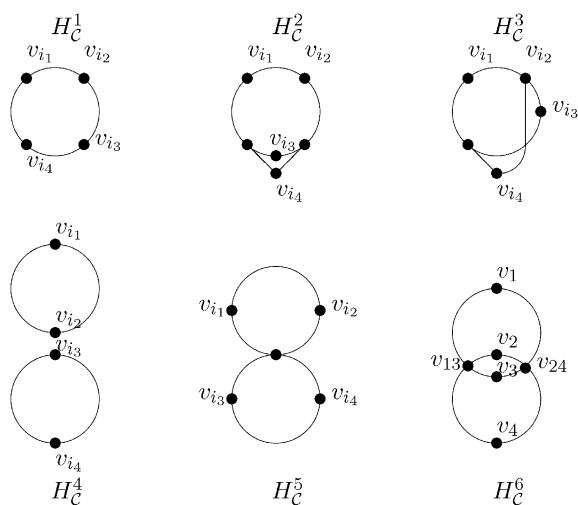
**Lemma 3.4.** *Let  $(X, C)$  be a pseudo 2-factor of a cubic 2-connected graph  $G$ . Suppose there is a degree-compatible subgraph  $H_C$  of  $G_C$  such that  $h(H)$  is  $C$ -compatible, and  $\chi'_h(H) = 3$ . Then  $G$  has a cycle double cover comprised of cycles from 5 circuits.*

**Proof.** Suppose  $H_C$  is a subgraph as specified in the Lemma. By Lemma 3.1 there is a collection of cycles  $\mathcal{C}_H$  belonging to 3 circuits which cover the edges of  $C$  once, and the edges of  $H \setminus \bigcup_{C \in \mathcal{C}} E(C)$  twice. Let  $H' = (G \setminus E(H) \cup X) \cup \bigcup_{v \in V(H_C) \setminus X} C(v)$ . In  $H'$  there is a 2-factor  $C'$  corresponding to  $C$ . Each cycle  $C' \in C'$  is such that  $h(C')$  is an even cycle. Consequently,  $\chi'_h(H') = 3$ . Thus, we can find 2 perfect matchings  $P_1$  and  $P_2$  in  $h(H')$  where  $P_1 \cup P_2 = \bigcup_{C' \in C'} E(h(C'))$ . For  $i = 1, 2$   $h(H') \setminus P_i$  is a disjoint union of cycles. Let  $\mathcal{C}_{H'}^i$  be the corresponding collection of cycles in  $H'$ . Then  $\mathcal{C}_{H'} = \mathcal{C}_{H'}^1 \cup \mathcal{C}_{H'}^2$  is a collection of cycles belonging to 2 circuits which cover the edges of  $C'$  once and the other edges of  $H'$  twice. It follows that  $\mathcal{C}_H \cup \mathcal{C}_{H'}$  is the desired cycle double cover of  $G$ .  $\square$

Let  $G$  be a 2-connected cubic graph and let  $(X, C)$  be a pseudo 2-factor. We suppose that, apart from loops,  $G_C$  is a 2-connected graph and has 4 odd vertices  $v_1, v_2, v_3, v_4$ . We wish to show that there exists a subgraph containing  $v_1, v_2, v_3, v_4$  which is one of the subgraphs illustrated in Fig. 9. In  $H_C^1$  there is a cycle containing all 4 vertices. In  $H_C^2$  and  $H_C^3$  there is a cycle containing exactly 3 of the vertices  $v_1, \dots, v_4$  which are denoted  $v_{i_1}, v_{i_2}$ , and  $v_{i_3}$ . There are 2 internally disjoint paths from the fourth vertex  $v_{i_4}$  to the cycle. In  $H_C^4$ , we have 2 disjoint cycles each containing exactly 2 of the vertices  $v_1, \dots, v_4$ . In  $H_C^5$  there are exactly 2 cycles meeting at one vertex, each cycle containing 2 of the vertices  $v_1, \dots, v_4$ . In  $H_C^6$ , there are 2 cycles meeting at 2 vertices (labelled  $v_{13}$  and  $v_{24}$ ) where each cycle contains 2 vertices of  $v_1, \dots, v_4$ .

**Lemma 3.5.** *Let  $G$  be a 2-connected loopless multigraph and let  $v_1, v_2, v_3, v_4$  be 4 vertices of  $G$ . The graph  $G$  has a subgraph  $H$  containing  $v_1, v_2, v_3, v_4$  where  $H$  is one of the graphs specified in Fig. 9. In (f), the vertices  $v_{13}$  and  $v_{24}$  form a 2-separating set in  $G$  which separates each pair of vertices  $v_i$  and  $v_j$ ,  $i, j = 1, 2, 3, 4$ .*

**Proof.** If there is a cycle in  $G$  containing  $v_1, v_2, v_3, v_4$ , then we have the subgraph  $H_C^1$  in Fig. 9. We may therefore assume that no such cycle exists. Suppose instead that there is a cycle  $C$  containing exactly 3 of the vertices, say  $v_{i_1}, v_{i_2}, v_{i_3}$ , where the remaining vertex

Fig. 9. Bridgeless subgraphs containing  $v_1, v_2, v_3, v_4$ .

$v_{i_4}$  lies outside of  $C$ . By Mengers theorem [2, p. 46], there are 2 internally vertex-disjoint paths from  $v_{i_4}$  to  $C$  which meet only at  $v_{i_4}$ . In this case, we have the two possibilities  $H_C^2$  and  $H_C^3$  illustrated in Fig. 9. We suppose now that  $G$  has no cycle containing 3 or 4 of the vertices  $v_i$ ,  $i = 1, 2, 3, 4$ . Since  $G$  is 2-connected, there is a cycle  $C$  containing  $v_1$  and  $v_2$  (but not  $v_3$  or  $v_4$ ). The cycle  $C$  is the union of 2 paths, say  $P_1$  and  $P_2$  from  $v_1$  to  $v_2$ . Since  $G$  is 2-connected, there are 2 internally disjoint paths  $P_3$  and  $P_4$  from  $v_3$  to  $C$  which meet only at  $v_3$ . Since it is assumed that  $G$  has no cycle containing 3 or more of the vertices  $v_i$ ,  $i = 1, 2, 3, 4$ , we may assume that  $P_3$  meets  $C$  along  $P_1$  at a vertex  $v_{13} \neq v_1, v_2$ . Similarly,  $P_4$  meets  $C$  along  $P_2$  at a vertex  $v_{24} \neq v_1, v_2$ . Let  $H = C \cup P_3 \cup P_4$ . We have that  $v_4 \notin V(H)$ , for otherwise there would be a cycle containing  $v_1, v_3$ , and  $v_4$  (given that  $v_3 \notin V(C)$ ). Again, by the 2-connectedness of  $G$ , there are 2 internally disjoint paths  $P_5$  and  $P_6$  from  $v_4$  to  $H$  which meet only at  $v_4$ . Depending on where  $P_5$  and  $P_6$  intersect  $H$ , the graph  $G$  must contain one of the subgraphs  $H_C^4$ ,  $H_C^5$  or  $H_C^6$  as illustrated in Fig. 9. In the case that  $G$  contains neither  $H_C^4$  nor  $H_C^5$ , it must be the case that the vertices  $v_{13}$  and  $v_{24}$  form a 2-separating set for each pair of vertices  $v_i, v_j$ ,  $i, j = 1, 2, 3, 4$ .  $\square$

**Lemma 3.6.** *Let  $G$  be a multigraph. There exists a forest  $F \subseteq G$  such that  $d_F(v) = d_G(v) \pmod{2}$ ,  $\forall v \in V(G)$ .*

**Proof.** By induction on the number of edges. If  $\varepsilon(G) = 0$ , then the lemma holds trivially. Suppose the lemma holds for all multigraphs having fewer than  $m$  edges ( $m > 0$ ), and suppose  $\varepsilon(G) = m$ . If  $G$  contains no cycles, then it is itself a forest and we can choose  $F = G$ . We suppose therefore that  $G$  contains a cycle  $C$ . Let  $G' = G \setminus E(C)$ . By assumption, there is a forest  $F \subseteq G'$  such that  $d_F(v) = d_{G'}(v) \pmod{2}$ ,  $\forall v \in V(G')$ . This means that

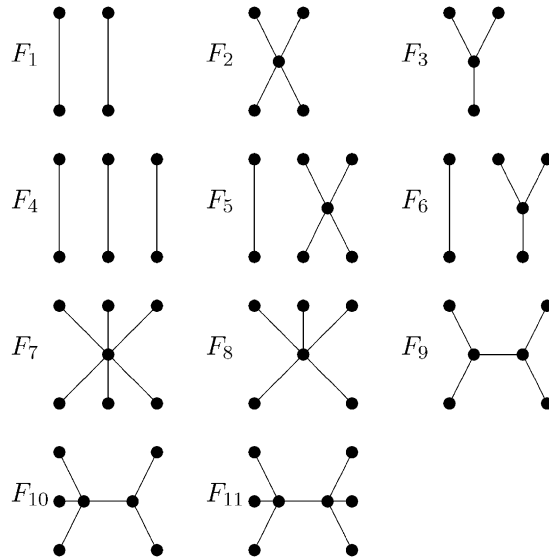


Fig. 10. Non-isomorphic, non-homeomorphic forests with 4 or 6 odd vertices.

$d_G(v) = d_F(v) \pmod{2} \forall v \in V(G)$ . Thus the lemma holds for all graphs with  $m$  edges, and the proof follows by induction.  $\square$

There are exactly 11 non-isomorphic, non-homeomorphic forests having 4 or 6 odd vertices. These are illustrated in Fig. 10.

Let  $G$  be a 2-connected cubic graph and let  $(X, \mathcal{C})$  be a pseudo 2-factor of  $G$ . We suppose that  $G_{\mathcal{C}}$  is 2-connected and has 4 odd vertices  $v_1, v_2, v_3, v_4$ . There is a bridgeless subgraph  $H'_{\mathcal{C}} \subseteq G_{\mathcal{C}}$  as in Lemma 3.5. The graph  $G'_{\mathcal{C}} = G_{\mathcal{C}} \setminus E(H'_{\mathcal{C}})$  has 4 or 6 odd vertices (depending on  $H'_{\mathcal{C}}$ ) and hence by Lemma 3.6 there is a forest  $F'_{\mathcal{C}} \subseteq G'_{\mathcal{C}}$  homeomorphic to one of the forests given in Fig. 10 where  $d_{F'_{\mathcal{C}}}(v) = d_{G'_{\mathcal{C}}}(v) \pmod{2} \forall v \in V(G_{\mathcal{C}})$ . Let  $H_{\mathcal{C}} = H'_{\mathcal{C}} \cup F'_{\mathcal{C}}$ . Then  $H_{\mathcal{C}}$  is a degree-compatible subgraph of  $G_{\mathcal{C}}$ .

For a multigraph  $M$ , we have a list of 9 conditions:

- (3.1.1)  $v_3(M) = 4$  and  $v_{\geq 5}(M) = 0$ .
- (3.1.2)  $v_3(M) = 4$ ,  $v_5(M) = 0$ ,  $v_6(M) = 1$ , and  $v_{\geq 7}(M) = 0$ .
- (3.1.3)  $v_3(M) = 3$ ,  $v_5(M) = 1$ ,  $v_{\geq 6}(M) = 0$ .
- (3.1.4)  $v_3(M) = 3$ ,  $v_5(M) = 1$ ,  $v_6(M) = 1$ ,  $v_{\geq 7}(M) = 0$ .
- (3.1.5)  $v_3(M) = 3$ ,  $v_5(M) = 0$ ,  $v_6(M) = 0$ ,  $v_7(M) = 1$ ,  $v_{\geq 8}(M) = 0$ .
- (3.1.6)  $v_3(M) = 2$ ,  $v_5(M) = 2$ ,  $v_{\geq 6}(M) = 0$ .
- (3.1.7)  $v_3(M) = 4$ ,  $v_5(M) = 0$ ,  $v_6(M) = 2$ ,  $v_{\geq 7}(M) = 0$ .
- (3.1.8)  $v_3(M) = 4$ ,  $v_5(M) = 0$ ,  $v_6(M) = 0$ ,  $v_7(M) = 0$ ,  $v_8(M) = 1$ , and  $v_{\geq 9}(M) = 0$ .
- (3.1.9)  $v_3(M) = 4$ ,  $v_5(M) = 0$ ,  $v_6(M) = 1$ ,  $v_7(M) = 0$ ,  $v_8(M) = 1$ , and  $v_{\geq 9}(M) = 0$ .

By considering all the possible subgraphs, the subgraph  $H_{\mathcal{C}}$  satisfies at least one of the 9 conditions listed above. The table below indicates for each combination of a forest from

Fig. 10 and a subgraph from Fig. 9 the subset of conditions which apply to  $H_C$ . In each case, at least one of these conditions must hold.

	$H_C^1$	$H_C^2$	$H_C^3$	$H_C^4$	$H_C^5$	$H_C^6$
$F_1$	(3.1.1)		(3.1.1) (3.1.3)	(3.1.1)	(3.1.1) (3.1.2)	(3.1.1) (3.1.2) (3.1.7)
$F_2$	(3.1.1) (3.1.2)		(3.1.1) (3.1.2) (3.1.5)	(3.1.1) (3.1.2)	(3.1.1) (3.1.2) (3.1.8)	(3.1.1) (3.1.2) (3.1.9)
$F_3$	(3.1.1)		(3.1.2) (3.1.3) (3.1.4) (3.1.6)	(3.1.3)	(3.1.3) (3.1.4)	Not possible
$F_4$		(3.1.1)				
$F_5$		(3.1.1) (3.1.2)				
$F_6$		(3.1.2) (3.1.3)				
$F_7$		(3.1.2) (3.1.8)				
$F_8$		(3.1.5) (3.1.8)				
$F_9$		(3.1.4) (3.1.6) (3.1.7)				
$F_{10}$		(3.1.2) (3.1.3) (3.1.4) (3.1.7)				
$F_{11}$		(3.1.2) (3.1.7)				

#### 4. Cycle covers

In this section, we prove some results on cycle coverings. Let  $G$  be a cubic graph and let  $p : E(G) \rightarrow \{0, 1, 2\}$  be a non-negative edge weighting of  $G$ . Let  $\mathcal{C}$  be a collection of cycles in  $G$ . For each edge  $e \in E(G)$  we let  $m_{\mathcal{C}}(e)$  be the number of cycles in  $\mathcal{C}$  containing  $e$ . We say that  $\mathcal{C}$  is a *cycle  $p$ -cover* for  $(G, p)$  if  $m_{\mathcal{C}}(e) = p(e)$ ,  $\forall e \in E(G)$ .

A weighting  $p : E(G) \rightarrow \mathbb{Z}^+$  is *eulerian* if  $\forall v \in V(G)$ ,  $\sum_{e \in E_v} p(e) = 0 \pmod{2}$ . For a weighted graph  $(G, p)$  with eulerian weighting  $p$  we define a *subdivision operation* where we subdivide an edge  $e_0$  with a vertex  $u$  and give the subdivided edges weight  $p(e_0)$ . Suppose we are given a weighted graph  $(G, p)$  and we perform a subdivision operation twice in succession, where we subdivide with vertices  $u$  and  $v$ . We then add an edge  $e$  of weight 2 between  $u$  and  $v$ . The resulting graph is  $G \odot (e)$ , and we denote the resulting (eulerian) weighting by  $p_{\odot(e)}$ .

Suppose we perform a subdivision operation 3 times, where we subdivide with vertices  $u_1, u_2$ , and  $u_3$ . We add a vertex  $v$  and join it to  $u_1, u_2$ , and  $u_3$  with edges of weight 2. The resulting graph is  $G \odot (v)$  and we denote the corresponding weighting by  $p_{\odot(v)}$ . We say that  $(G \odot (e), p_{\odot(e)})$  (resp.,  $(G \odot (v), p_{\odot(v)})$ ) preserves cycle coverings if, given  $(G, p)$  has a cycle  $p$ -cover, then  $(G \odot (e), p_{\odot(e)})$  has a cycle  $p_{\odot(e)}$ -cover (resp.,  $(G \odot (v), p_{\odot(v)})$  has a cycle  $p_{\odot(v)}$ -cover. Similarly, we say that an insertion operation preserves 3-edge colourings if, given  $G$  is 3-edge colourable, the graph resulting from  $G$  after the insertion operation is also 3-edge colourable.

We define the *distance* between two edges  $e_0$  and  $e_1$  in a connected graph  $G$  to be the number of edges in the shortest path containing  $e_0$  and  $e_1$  minus 1. This distance we denote by  $\text{dist}_G(e_0, e_1)$ .

**Theorem 4.1.** *Let  $(G, p)$  be a weighted cubic graph where  $p : E(G) \rightarrow \{1, 2\}$  and  $p$  is eulerian.*

- (i) *Let  $G' = G \odot (e)$  and  $p' = p_{\odot(e)}$  where  $e$  has endvertices in edges  $e_0$  and  $e_1$  in  $G$ . If  $\text{dist}_G(e_0, e_1) \leq 2$ , then  $(G', p')$  preserves cycle coverings, and  $G'$  preserves 3-edge colourings. Consequently, if  $e_0$  and  $e_1$  belong to a 5-cycle, then  $(G', p')$  preserves cycle coverings, and  $G'$  preserves 3-edge colourings.*
- (ii) *Let  $G' = G \odot (v)$  and  $p' = p_{\odot(v)}$  where  $v$  has neighbours inserted in the edges  $e_0, e_1$ , and  $e_2$  in  $G$ . If  $e_0, e_1$ , and  $e_2$  belong to a cycle of length at most 5 in  $G$ , then  $(G', p')$  preserves cycle coverings, and  $G'$  preserves 3-edge colourings.*

**Proof.** (i) Let  $(G', p')$  and  $e_0, e_1$  be as in (i). Suppose  $\text{dist}_G(e_0, e_1) = 0$ ; that is,  $e_0 = e_1$ . Let  $C \in \mathcal{C}$  be a cycle containing  $e_0$ . Then  $e$  is a chord of  $C$  in  $G'$  and we can replace  $C$  by 2 cycles  $C_1, C_2 \subset C \cup \{e\}$  where  $C_1$  and  $C_2$  cover  $e$  twice and  $C$  once. It then follows that  $C' = (C \setminus \{C\}) \cup \{C_1, C_2\}$  is a cycle  $p'$ -cover of  $(G', p')$ . We also see that  $G$  can be obtained from  $G'$  via an  $\circ$ -reduction. Thus  $\chi'(G) = \chi'(G')$  and  $G'$  preserves 3-edge colourings.

Suppose that  $\text{dist}_G(e_0, e_1) = 1$ ; that is,  $e_0$  and  $e_1$  are incident with a common vertex. Let  $C_0, C_1 \in \mathcal{C}$  be cycles where  $C_0$  contains  $e_0$  and  $C_1$  contains  $e_1$ . If  $e_1 \in E(C_0)$ , then  $e$  is a chord of  $C_0$  and we may adopt the previous argument. So we may assume  $e_1 \notin E(C_0)$  and likewise,  $e_1 \notin E(C_1)$ . Let  $H = h(C_0 \cup C_1)$ . We have that  $\chi'(H) = 3$ , as  $C_0 \nabla C_1$  corresponds to a 2-factor with even cycles in  $H$ . Moreover, we see that  $e$  is a chord of some cycle in  $C_0 \nabla C_1$ , and consequently  $H' = h(C_0 \cup C_1 \cup \{e\})$  is also 3-edge colourable. By Lemma 3.1 there is a collection of cycles  $\mathcal{C}_{H'}$  in  $H'$  which covers  $C_0 \nabla C_1$  once, and the other edges of  $H'$  twice. Let  $\mathcal{D}$  be the collection of cycles of  $C_0 \cup C_1 \cup \{e\}$  corresponding to  $\mathcal{C}_{H'}$ . Then  $C' = (C \setminus \{C_0, C_1\}) \cup \mathcal{D}$  is seen to be a cycle  $p'$ -cover for  $(G', p')$ . We note that  $G$  can be obtained from  $G'$  via a  $\Delta$ -reduction and consequently  $\chi'(G) = \chi'(G')$ . Thus  $G'$  preserves 3-edge colourings.

We suppose now that  $\text{dist}_G(e_0, e_1) = 2$ . Let  $C_0, C_1 \in \mathcal{C}$  where  $e_0 \in E(C_0)$  and  $e_1 \in E(C_1)$ . We may assume that  $e_0 \notin E(C_1)$ ,  $e_1 \notin E(C_0)$ , and there is an edge  $e_{01} \in E(G)$  lying on a path of length 3 between  $e_0$  and  $e_1$ . We will consider 2 cases:

*Case 1:*  $E(C_0) \cap E(C_1) = \emptyset$ .

Let  $C_{01} \in \mathcal{C}$  be a cycle containing  $e_{01}$ . We may assume  $e_0, e_1 \notin E(C_{01})$ , for otherwise we can jump ahead to the second case. Let  $H = h(C_0 \cup C_1 \cup C_{01})$ . We have that  $\chi'(H) = 3$

and moreover,  $e$  is a chord of a cycle in the 2-factor of  $H \ C_0 \nabla C_1 \nabla C_{01}$ . We can now apply the previous argument to obtain a cycle  $p'$ -cover for  $G'$ .

Case 2:  $E(C_0) \cap E(C_1) \neq \emptyset$ .

Consider  $H = h(C_0 \cup C_1)$ . Suppose  $e = xy$ . We shall assume that  $\mathcal{C}$  is a cycle cover having a maximum number of cycles. Let  $C_0^h$  and  $C_1^h$  be the cycles of  $H$  corresponding to  $C_0$  and  $C_1$ , respectively. We have that  $\chi'(H) = 3$  and  $C_0^h \nabla C_1^h$  is a 2-factor (with even cycles). If  $e$  is a chord of some cycle in  $C_0 \nabla C_1$ , then we proceed as before. So we may assume that  $e$  lies between 2 cycles of  $C_0^h \nabla C_1^h$ . Colour the edges of the cycles of  $C_0^h \nabla C_1^h$  alternatively with colours green and blue in such a way that the edges containing  $e_0$  and  $e_1$  are given the same colour, say green. Colour the remaining edges of  $H$  red. Let  $D_{gr}^h$  and  $D_{br}^h$  be the circuits induced by the green–red and blue–red edges, respectively. Let  $\mathcal{D}_{gr}^h$  and  $\mathcal{D}_{br}^h$  be the set of cycles in  $D_{gr}^h$  and  $D_{br}^h$ , respectively. We let  $\mathcal{D}_{gr}$  and  $\mathcal{D}_{br}$  be the sets of cycles in  $G$  corresponding to  $\mathcal{D}_{gr}^h$  and  $\mathcal{D}_{br}^h$ , respectively. Similarly, we let  $\mathcal{D}'_{gr}$  and  $\mathcal{D}'_{br}$  be the sets of cycles in  $G'$  corresponding to  $\mathcal{D}_{gr}$  and  $\mathcal{D}_{br}$ , and we let  $\mathcal{D}'_{gr}$  and  $\mathcal{D}'_{br}$  be the circuits of  $G'$  corresponding to  $\mathcal{D}_{gr}$  and  $\mathcal{D}_{br}$ . If  $|\mathcal{D}_{gr}^h| > 1$ , or  $|\mathcal{D}_{br}^h| > 1$ , then  $(\mathcal{C} \setminus \{C_0, C_1\}) \cup \mathcal{D}_{gr} \cup \mathcal{D}_{br}$  would be a cycle  $p$ -cover of  $(G, p)$  with more cycles than  $\mathcal{C}$ , contradicting the maximality of  $\mathcal{C}$ . Thus both  $\mathcal{D}_{gr}$  and  $\mathcal{D}_{br}$  are cycles. This means that  $e$  is a chord of  $\mathcal{D}'_{gr}$  in  $G'$ , and we can split  $\mathcal{D}'_{gr} \cup \{e\}$  into 2 cycles  $\mathcal{D}_{gr}^1$  and  $\mathcal{D}_{gr}^2$  where  $\{\mathcal{D}_{gr}^1, \mathcal{D}_{gr}^2\}$  cover  $\mathcal{D}'_{gr}$  once and  $e$  twice. Let  $\mathcal{C}' = (\mathcal{C} \setminus \{C_0, C_1\}) \cup \{\mathcal{D}_{gr}^1, \mathcal{D}_{gr}^2, \mathcal{D}'_{br}\}$ . Then  $\mathcal{C}'$  is a cycle  $p'$ -cover for  $(G', p')$ .

To show that  $G'$  preserves 3-edge colourings, we first note that a cubic graph is 3-edge colourable iff it has three 2-factors which form a double cycle cover. Suppose  $\chi'(G) = 3$ , and let  $\mathcal{C}$  be a double cover consisting of cycles from three 2-factors. We may assume  $C_0$  and  $C_1$  are disjoint (as in case 1) or  $C_0$  and  $C_1$  are the same cycle. Let  $D$  be the 4-cycle in  $G'$  containing  $e$  and  $e_{01}$ . Let  $\mathcal{C}' = (\mathcal{C} \setminus \{C_0, C_1, C_{01}\}) \cup \{(C_0 \cup C_1) \nabla D, C_{01} \nabla D\}$ . Now  $\mathcal{C}'$  is seen to be a cycle  $p'$ -cover of  $G'$ , and  $\mathcal{C}'$  is a union of three 2-factors. Thus  $\chi'(G') = 3$ . This shows that  $G'$  preserves 3-edge colourings. This proves (i).

To prove (ii) we note that one can obtain  $(G', p')$  by performing an edge insertion operation twice, each time inserting an edge which is a chord of a 5-cycle. The proof then follows by (i).  $\square$

**Lemma 4.2.** *Let  $(G, p)$  be a weighted cubic graph where  $p : E(G) \rightarrow \{1, 2\}$  is an eulerian weighting. Let  $C$  be a chordless cycle of  $G$  where  $p(e) = 1$ ,  $e \in E(C)$ . Suppose that  $G$  is the union of subgraphs  $H_i$ ,  $i = 1, \dots, t$   $t \leq 4$  which intersect along  $C$ . For  $i = 1, \dots, t$  let  $p_i$  be the weighting  $p$  restricted to  $H_i$ .*

- If  $d_{H_i}(C) = 3 \ \forall i$  and each  $(H_i, p_i)$  has a cycle  $p_i$ -cover, then  $(G, p)$  has a cycle  $p$ -cover.*
- Suppose  $t = 2$  and  $d_{H_1}(C) \leq 5$ , and  $d_{H_2}(C) \leq 3$ . If for  $i = 1, 2$  each  $(H_i, p_i)$  has a cycle  $p_i$ -cover, then  $(G, p)$  has a cycle  $p$ -cover. Moreover, if  $\chi'_h(H_1) = \chi'_h(H_2) = 3$ , then  $\chi'(G) = 3$ .*

**Proof.** We shall first prove (a). For  $i = 1, \dots, t$  let  $H'_i$  be the graph obtained from  $H_i$  by contracting  $C$  into a single vertex, and we let  $p'_i$  be a weighting of  $H'_i$  where  $p'_i$  is the same as  $p_i$  restricted to  $H_i \setminus E(C)$ . Given each  $(H_i, p_i)$  has a cycle  $p_i$ -cover, we have that each



$(H'_i, p'_i)$  has a cycle  $p'_i$ -cover, say  $\mathcal{D}'_i$ . We form a cubic graph  $H'$  from  $G$  by contracting each graph  $H_i \setminus C$  into a single vertex, so that  $H'$  consists of  $C$  together with  $t$  independent vertices joined to  $C$ . Let  $q'$  be a weighting of  $H'$  where  $q'(e) = 1, \forall e \in E(C)$  and  $q'(e) = 2, \forall e \notin E(C)$ . According to Example 2,  $(H', q')$  has a cycle  $q'$ -cover, say  $\mathcal{D}'$ . We can now splice together the cycle collections  $\mathcal{D}'_i, i = 1, \dots, t$  with  $\mathcal{D}'$  to obtain a cycle collection  $\mathcal{D}$  which is a cycle  $p$ -cover for  $G$ .

To prove (b) let  $H'_1$  be the cubic graph obtained from  $G$  by contracting  $H_2 \setminus C$  into single vertex  $u_1$ , and if necessary, performing a  $\vee$ -reduction on  $u_1$  if it has degree 2. We define a weighting  $p'_1$  on  $H'_1$  where  $p'_1(e) = p_1(e) \forall e \in E(H_1)$  and  $p_1(e) = 2$  for all other edges. Let  $H'_2$  be the graph obtained from  $H_2$  by contracting  $C$  into a single vertex  $u_2$ , performing a  $\vee$ -reduction on  $u_2$  if it has degree 2. We define a weighting  $p'_2$  on  $H'_2$  where  $p'_2$  is the same  $p_2$  restricted to  $H_2 \setminus E(C)$ . One obtains  $(H'_1, p'_1)$  from  $H_1$  by either inserting a vertex of degree 3 in  $C$ , or inserting a chord in  $C$ . Assuming  $(H_1, p_1)$  has a cycle  $p_1$ -cover, we observe that  $d_{H_1}(C) \leq 5$ , and thus Theorem 4.1 (ii) implies that  $(H'_1, p'_1)$  has a cycle  $p'_1$ -cover, say  $\mathcal{D}'_1$ . Assuming  $(H_2, p_2)$  has a cycle  $p_2$ -cover, we have that  $(H'_2, p'_2)$  has a cycle  $p'_2$ -cover, say  $\mathcal{D}'_2$ . We now splice together  $\mathcal{D}'_1$  and  $\mathcal{D}'_2$  to obtain a cycle  $p$ -cover for  $(G, p)$ .

If we assume that  $H_1$  and  $H_2$  are 3-edge colourable, then  $H'_1$  is 3-edge colourable (by Theorem 4.1 (ii)) and  $H'_2$  is 3-edge colourable. Since  $G$  is obtained from  $H_1$  and  $H_2$ , either via a (3-chromatic) vertex expansion  $u \rightarrow H'_2$  or via a 2-edge expansion, the graph  $G$  is 3-edge colourable.  $\square$

## 5. K-joins

For a positive integer  $k > 0$ , we define a  $k$ -join of 2 graphs  $G$  and  $H$  where we join  $G$  and  $H$  by taking  $k$  vertices  $g_1, g_2, \dots, g_k$  in  $G$  and  $k$  vertices  $h_1, h_2, \dots, h_k$  in  $H$  and identify each pair of vertices  $g_i, h_i, i = 1, 2, \dots, k$  with single vertices. We denote the resulting graph by  $(G; g_1, \dots, g_k) \vee_k (H; h_1, \dots, h_k)$ . We define the 0-join of  $G$  and  $H$  to be the disjoint union of  $G$  and  $H$ , and denote this graph by  $G \vee_0 H$ . A  $k$ -join is said to be *odd* (resp., *even*) if  $d_G(g_i)$  and  $d_H(h_i)$  are odd (resp., even) for all  $i$ . Here, we use the symbol  $\vee_k^o$  (resp.,  $\vee_k^e$ ) in place of  $\vee_k$  to denote an odd (resp., even)  $k$ -join.

If  $d_G(g_i)$  is even (resp., odd) for all  $i$  and  $d_H(h_i)$  is odd (resp., even) for all  $i$ , then the  $k$ -join is said to be *even-odd* (resp., *odd-even*). We use the symbol  $\vee_k^{eo}$  (resp.,  $\vee_k^{oe}$ ) in place of  $\vee_k$  to denote an even-odd (resp., odd-even)  $k$ -join.

For two families of graphs  $\mathcal{G}$  and  $\mathcal{H}$  where each graph has at least  $k$  vertices, we define  $\mathcal{G} \vee_k \mathcal{H}$  to be the set of  $k$ -joins of graphs in  $\mathcal{G}$  with graphs in  $\mathcal{H}$ . We define  $\mathcal{G} \vee_k^o \mathcal{H}$  (resp.,  $\mathcal{G} \vee_k^e \mathcal{H}$ ) to be the set of odd (resp., even)  $k$ -joins of graphs from  $\mathcal{G}$  and  $\mathcal{H}$ . In a similar fashion, we define  $\mathcal{G} \vee_k^{oe} \mathcal{H}$  and  $\mathcal{G} \vee_k^{eo} \mathcal{H}$ .

We define  $(\mathcal{G})_k^1 = \mathcal{G}$ , and for  $i = 2, 3, \dots$  we define  $(\mathcal{G})_k^i = (\mathcal{G})_k^{i-1} \vee_k \mathcal{G}$ . We let  $(\mathcal{G})_k = \bigcup_{i \geq 1} (\mathcal{G})_k^i$ , and define  $(\mathcal{G})_k^{i,e}$  (resp.,  $(\mathcal{G})_k^{i,o}$ ) in a similar fashion, replacing the symbol  $\vee_k$  with the symbol  $\vee_k^e$  (resp.,  $\vee_k^o$ ) in the previous definition.

For collections of graphs  $\mathcal{G}_1, \dots, \mathcal{G}_n$  we define a sequence of  $k$ -joins  $\mathcal{G}_1 \vee_k \dots \vee_k \mathcal{G}_n$  recursively by

$$\mathcal{G}_1 \vee_k \dots \vee_k \mathcal{G}_n = (\mathcal{G}_1 \vee_k \dots \vee_k \mathcal{G}_{n-1}) \vee_k \mathcal{G}_n.$$

We define  $\mathcal{G}_1 \vee_k^o \dots \vee_k^o \mathcal{G}_n$  and  $\mathcal{G}_1 \vee_k^e \dots \vee_k^e \mathcal{G}_n$  similarly.

Let  $\mathcal{F}_2$  be the family of graphs consisting of graphs which are the edge-disjoint union of a cycle and a path, the path going between 2 vertices on the cycle. Each such graph has exactly 2 odd vertices (having degree 3). Let  $\mathcal{F}_4$  be the family of bridgeless graphs with exactly 4 odd vertices  $v_1, v_2, v_3, v_4$ , being the union of a graph containing  $v_1, \dots, v_4$  as in Fig. 9, and a tree homeomorphic to one in Fig. 10. Let  $\mathcal{F}_2^* = \bigcup_{i \geq 1} (\mathcal{F}_2)_{i,1}^{i,0}$ . Each graph  $F \in \mathcal{F}_2^*$  is a block chain whose blocks belong to  $\mathcal{F}_2$ . Moreover, each  $F \in \mathcal{F}_2^*$  has exactly 2 odd vertices (having degree 3), one in each of its endblocks. Let

$$\begin{aligned} \mathcal{F}_4^* = & \mathcal{F}_4 \cup (\mathcal{F}_4 \vee_1^o \mathcal{F}_2^*) \cup (\mathcal{F}_4 \vee_1^o \mathcal{F}_2^* \vee_1^o \mathcal{F}_2^*) \cup (\mathcal{F}_4 \vee_1^o \mathcal{F}_2^* \vee_1^o \mathcal{F}_2^* \vee_1^o \mathcal{F}_2^*) \\ & \cup (\mathcal{F}_4 \vee_1^o \mathcal{F}_2^* \vee_1^o \mathcal{F}_2^* \vee_1^o \mathcal{F}_2^* \vee_1^o \mathcal{F}_2^*). \end{aligned}$$

Each member of  $\mathcal{F}_4^*$  consists of a graph  $G \in \mathcal{F}_4$  with block chains from  $\mathcal{F}_2^*$  joined via an odd 1-join to some or none of the odd vertices of  $G$ .

**Lemma 5.1.** *Let  $G$  be a 2-edge connected graph having exactly 2 odd vertices  $v_1$  and  $v_2$ . Then  $G$  contains a subgraph  $H \in \mathcal{F}_2^*$  whose odd vertices are exactly  $v_1$  and  $v_2$ .*

**Proof.** Suppose  $v_1$  and  $v_2$  belong to the same block  $B$  of  $G$ . Then there is a cycle  $C$  in  $B$  containing  $v_1$  and  $v_2$ . Let  $G' = G \setminus E(C)$ . Then  $v_1$  and  $v_2$  are exactly the odd vertices of  $G'$ . They must belong to the same component in  $G'$ , and consequently, there must be a path  $P$  in  $G'$  between them. Let  $H = C \cup P$ . Then  $H \in \mathcal{F}_2$  (hence  $H \in \mathcal{F}_2^*$ ) and moreover,  $v_1$  and  $v_2$  are exactly the odd vertices of  $H$ .

Suppose now that  $v_1$  and  $v_2$  belong to different blocks of  $G$ . Then there is a block chain  $B_0 \cdots B_k$  where  $v_1 \in V(B_0)$ ,  $v_2 \in V(B_k)$ , and  $v_1, v_2 \notin V(B_i)$  for  $0 < i < k$ . Let  $V(B_i) \cap V(B_{i+1}) = \{u_{i+1}\}$ ,  $i = 0, \dots, k-1$ , and let  $u_0 = v_1$ , and  $u_{k+1} = v_2$ . Since  $d_{B_0}(u_0)$  is odd, it follows that  $d_{B_0}(u_1)$  is odd and thus  $d_{B_{i-1}}(u_i)$ , and  $d_{B_i}(u_i)$  are odd for  $i = 1, \dots, k$ . Since each  $B_i$ ,  $i = 0, \dots, k$  is 2-connected (and is not a single edge), there are subgraphs  $H_i \subseteq B_i$ ,  $i = 0, \dots, k$  where  $H_i \in \mathcal{F}_2$  and  $u_i$  and  $u_{i+1}$  are exactly the odd vertices of  $H_i$ . Let  $H = H_0 \cup \dots \cup H_k$ . Then  $H \in \mathcal{F}_2^*$ , and  $v_1$  and  $v_2$  are exactly the odd vertices of  $H$ .

Let  $G$  be a 2-connected cubic graph and let  $(X, C)$  be a pseudo 2-factor of  $G$ .

**Proposition 5.2.** *Let  $H_C = (H_1)_C \cup (H_2)_C$  be a loopless subgraph of  $G_C$  where  $(H_1)_C$  intersects  $(H_2)_C$  at exactly one vertex  $v$ .*

- (i) *If  $h(H_1)$  and  $h(H_2)$  are  $\mathcal{C}$ -compatible,  $d_{(H_1)_C}(v) \leq 5$ , and  $d_{(H_2)_C}(v) \leq 3$ , then  $h(H)$  is  $\mathcal{C}$ -compatible. Moreover, if  $\chi'_h(H_1) = \chi'_h(H_2) = 3$ , then  $\chi'_h(H) = 3$ .*
- (ii) *If  $H_C \in \mathcal{F}_2^*$ , then  $\chi'_h(H) = 3$ .*

**Proof.** To prove (i) we first note that  $H_1$  intersects  $H_2$  along the cycle  $C = C(v)$  which has no chords in  $H_1 \cup H_2$  (since  $H_C$  is assumed to be loopless). We suppose that  $h(H_1)$  and  $h(H_2)$  are  $\mathcal{C}$ -compatible,  $d_{(H_1)_C}(v) \leq 5$ , and  $d_{(H_2)_C}(v) \leq 3$ . We have that  $(h(H_i), p_{h(H_i)})$  has a cycle  $p_{h(H_i)}$ -cover for  $i = 1, 2$ , and  $d_{H_1}(C) \leq 5$ , and  $d_{H_2}(C) \leq 3$ . Now Lemma 4.2 b) implies that  $(h(H), p_{h(H)})$  has a cycle  $p_{h(H)}$ -cover, and consequently  $h(H)$  is  $\mathcal{C}$ -compatible. Moreover, if  $\chi'_h(H_1) = \chi'_h(H_2) = 3$ , then  $\chi'_h(H) = 3$ .

To prove (ii) suppose that  $H_C \in \mathcal{F}_2^*$ . If  $H_C \in \mathcal{F}_2$ , then we can reduce  $h(H)$  to a multiple 3-edge via  $\circ$ -,  $\Delta$ -, and  $\square$ -reductions. In this case,  $\chi'_h(H) = 3$ . We suppose therefore that  $H_C \in \mathcal{F}_2^* \setminus \mathcal{F}_2$ . Then  $H_C = (H_1)_C \cup (H_2)_C$  where  $(H_1)_C, (H_2)_C \in \mathcal{F}_2^*$  and  $(H_1)_C$  intersects  $(H_2)_C$  at exactly one vertex, say  $v$ , where  $d_{(H_i)_C}(v) = 3$ ,  $i = 1, 2$ . We may assume that  $\chi'(h(H_1)) = \chi'(h(H_2)) = 3$ . It now follows from (i) that  $\chi'_h(H) = 3$ .

## 6. Proof of the main theorem

In this section, we give a proof of Theorem 1.2. Let  $G$  be a 2-connected cubic graph with  $o(G) \leq 4$  and let  $(X, \mathcal{C})$  be a pseudo 2-factor of  $G$  where  $|X|$  plus the number of odd cycles in  $\mathcal{C}$  is at most 4. Let  $G_C$  be the graph obtained from  $G$  by contracting the cycles of  $\mathcal{C}$ .

**Theorem 6.1.** *Either the graph  $G_C$  contains a degree-compatible subgraph  $H_C$  such that  $h(H)$  is  $\mathcal{C}$ -compatible, or it contains a non-trivial 3-edge cut.*

**Proof.** If  $o(G) = 0$ , then  $\chi'(G) = 3$  and result holds taking  $H_C = G_C$ . If  $o(G) = 2$ , then by Lemma 5.1 there is a degree compatible subgraph  $H_C$  of  $G_C$  belonging to  $\mathcal{F}_2^*$ . By Proposition 5.2 (ii),  $\chi'_h(H) = 3$ , and consequently  $h(H)$  is  $\mathcal{C}$ -compatible. Thus we may assume that  $o(G) = 4$ , and  $v_1, v_2, v_3, v_4$  are the odd vertices of  $G_C$ .

*Case 1:* Suppose  $G_C$  has a block  $B_C$  containing all 4 of the odd vertices  $v_1, v_2, v_3, v_4$ .

The vertices  $v_1, \dots, v_4$  are easily seen to be the odd vertices of  $B_C$ . According to Lemmas 3.5 and 3.6, there is a (loopless) subgraph  $H_C \subseteq B_C$  where  $H_C \in \mathcal{F}_4$  and  $v_1, \dots, v_4$  are exactly the odd vertices of  $H_C$ . For  $v \in V(H_C) \setminus X$  let  $C'(v)$  be the cycle in  $h(H)$  corresponding to  $C(v)$  (ie.  $C_{h(H)}(v)$ ).

For each 2- or 3-cycle  $C'(v)$ ,  $v \in V(H_C) \setminus X$ , we perform  $\circ$ - and  $\Delta$ -reductions, respectively. Next we perform  $\square$ -reductions on all 4-cycles  $C'(v)$ , and this we do in such a way that the resulting (cubic) graph  $h(H)'$  is bridgeless (this is possible by Lemma 1.7). Here, is an overview of the notation to be used in the ensuing proof.

$G_C$ : graph obtained from  $G$  by contracting cycles of  $\mathcal{C}$ .

$H_C$ : degree compatible subgraph of  $B_C$  belonging to  $\mathcal{F}_4$ .

$H$ : subgraph in  $G$  corresponding to  $H_C$ .

$h(H)$ : cubic graph homeomorphic to  $H$ .

$h(H)'$ : bridgeless graph obtained from  $h(H)$  via  $\circ$ -,  $\Delta$ -, and  $\square$ -reductions.

$C(v)$ : cycle in  $\mathcal{C}$  corresponding to  $v \in V(G_C) \setminus X$ .

$C'(v)$ : cycle in  $h(H)$  corresponding to  $C(v)$ .

$G'_C$ : the graph  $G_C \setminus E(H_C)$ .

$G'$ : subgraph in  $G$  corresponding to  $G'_C$ .

We know that exactly one of the conditions (3.1.1)–(3.1.9) holds for  $H_C$ . We shall examine two subcases:

*Case 1.1:* Suppose  $H_C$  satisfies one of (3.1.1)–(3.1.8).

We have that  $v(h(H)') \leq 16$ . If  $\chi'(h(H)') = 3$ , then  $\chi'(h(H)) = 3$  (by Lemma 1.7 and Corollary 2.2). It then follows from Lemma 3.1 that  $h(H)$  is  $\mathcal{C}$ -compatible. Thus we may assume that  $\chi'(h(H)') = \chi'(h(H)) = 4$ . Since  $v(h(H)') \leq 16$ , Proposition 2.7 implies that  $h(H)'$  is a 3-chromatic expansion of  $P_{10}$  and consequently  $h(H)$  is a 3-chromatic expansion of  $P_{10}$ . Let  $v \rightarrow A_v$ ,  $v \in P_{10}$  be a representation of this expansion. For each  $A_v$ ,  $v \in P_{10}$  let  $A'_v$  be the subgraph of  $h(H)$  induced by the edges corresponding to those in  $A_v$ . If  $H_C$  satisfies one of (3.1.1)–(3.1.3), (3.1.5), or (3.1.8), then all but at most one of the cycles  $C'(v)$ ,  $v \in V(H_C)$  belongs to some  $A'_v$ ,  $v \in V(P_{10})$ . In this case, Lemma 3.2 implies that there is a collection of cycles in  $h(H)$  covering each of the cycles  $C'(v)$ ,  $v \in V(H_C)$  once, and the other edges of  $h(H)$  twice. This means that  $h(H)$  is  $\mathcal{C}$ -compatible.

We suppose that  $H_C$  satisfies exactly one of (3.1.4), (3.1.6), or (3.1.7) and exactly 2 of the cycles  $C'(v)$ ,  $v \in V(H_C)$  say  $C'(u_1)$  and  $C'(u_2)$ , do not belong to any  $A'_v$ . We may assume that  $C'(u_1)$  intersects exactly 5 of the subgraphs  $A'_v$  and  $C'(u_2)$  intersects the other 5  $A'_v$ 's; that is, they correspond to 2 vertex-disjoint 5-cycles of  $P_{10}$ . Thus  $h(H) \setminus E(C'(u_1) \cup C'(u_2))$  has at least 5 components. However, since we are given that  $H_C$  is the union of 2 graphs, one from each of Figs. 9 and 10, and  $H_C$  satisfies one of (3.1.4), (3.1.6), or (3.1.7), one sees that  $h(H) \setminus (C'(u_1) \cup C'(u_2))$  can have at most 4 components. This yields a contradiction, and this concludes the proof for case 1.1.

*Case 1.2:* Suppose  $B_C$  contains no degree-compatible subgraph in  $\mathcal{F}_4$  which satisfies one of (3.1.1)–(3.1.8).

By Theorem A.1 in the Appendix A, either  $B_C$  contains a degree-compatible subgraph  $H_C$  which is  $\mathcal{C}$ -compatible, or  $G$  has a non-trivial 3-edge cut which separates a vertex of  $X$  or odd cycle of  $\mathcal{C}$  in  $B$  from the other vertices of  $X$  or odd cycles in  $\mathcal{C}$  in  $B$ . In this case, the theorem is seen to hold.

*Case 2:* Suppose no block of  $G_C$  contains  $v_1, v_2, v_3, v_4$ .

We shall divide this case into 2 subcases:

*Case 2.1:* There is a block  $B_C \subseteq G_C$  having 4 odd vertices.

We may assume that  $B_C$  has odd vertices  $u_1, u_2, u_3, u_4$ . For each  $u_i$  which is odd in  $G_C$  we may assume  $u_i = v_i$ . If  $u_i$  is not odd in  $G_C$ , we may assume there is a block chain  $(B_i)_C = (B_{i0})_C \cup \dots \cup (B_{ir_i})_C$  where  $u_i \in V((B_{i0})_C)$ ,  $v_i \in V((B_{ir_i})_C)$  and  $u_i$  and  $v_i$  are exactly the odd vertices of the chain. Let  $G_C^1$  be the subgraph obtained from  $G_C$  where for each  $i = 1, \dots, 4$  we delete all the vertices of  $(B_i)_C$  except  $u_i$ . Now  $u_1, \dots, u_4$  are seen to be the odd vertices of  $G_C^1$  which belong to the block  $B_C$ . If  $G^1$  has a non-trivial 3-edge cut which separates a vertex or odd cycle corresponding to one of the vertices  $u_1, \dots, u_4$ , then such a cut will also be a non-trivial 3-edge cut of  $G$ . So we may assume that no such cuts exist in  $G^1$ . Now according to Theorem A.1, there is a degree-compatible subgraph  $J_C$  for which  $h(J)$  is  $\mathcal{C}$ -compatible and one of two things hold: either  $J_C \in \mathcal{F}_4$  and one of (3.1.1)–(3.1.8) holds, or every odd degree vertex of  $J_C$  has degree three.

According to Lemma 5.1 the chain  $(B_i)_C$  contains a subgraph  $(H_i)_C \in \mathcal{F}_2^*$  whose odd vertices are exactly  $u_i$  and  $v_i$ . If  $u_i = v_i$ , then we let  $(H_i)_C = u_i = v_i$ . Let  $H_C = J_C \cup \bigcup_i (H_i)_C$ . We have that  $H_C \in \mathcal{F}_4^*$ .

By assumption,  $u_i$ ,  $i = 1, \dots, 4$  cannot all be odd in  $G_C$ . We may therefore assume that at least one of the  $u_i$ 's, say  $u_1$ , is not odd in  $G_C$ . Suppose first that  $d_{J_C}(u_i) \leq 5$ , for  $i = 1, \dots, 4$ . For each  $i$  where  $u_i \neq v_i$  we have  $h(B_i)$  is  $\mathcal{C}$ -compatible since  $\chi'_h(B_i) = 3$

by Proposition 5.2 (ii). It follows from repeated application of Proposition 5.2(i) that  $h(H)$  is  $\mathcal{C}$ -compatible. As such we can assume that  $J_{\mathcal{C}}$  has odd vertices of degree at least 7. This means that  $J_{\mathcal{C}}$  must satisfy one of (3.1.1)–(3.1.8), and in particular, it must satisfy (3.1.5). Thus  $J_{\mathcal{C}}$  has one vertex of degree 7, and 3 vertices of degree 3. If for some  $i$ ,  $u_i = v_i$ , and  $d_{J_{\mathcal{C}}}(u_i) = 7$ , then  $h(H)$  is  $\mathcal{C}$ -compatible by Proposition 5.2(i). Thus we may assume that  $d_{J_{\mathcal{C}}}(u_i) = 7$  for some  $u_i \neq v_i$ , and this we can assume this holds for  $u_1$  (and  $d_{J_{\mathcal{C}}}(u_i) = 3$ ,  $i = 2, 3, 4$ ).

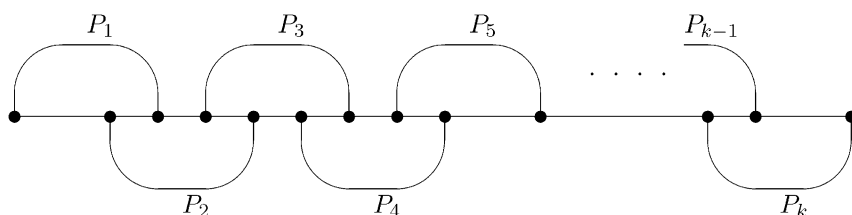
Let  $J_1$  be the graph obtained from  $J \cup H_1$  where we contract  $H_1 \setminus C(u_1)$  into a single vertex  $w_1$ . We can reduce each 2-, 3-, and 4-cycle  $C_{h(J)}(v) \subseteq h(J_1)$  via  $\circ$ -,  $\Delta$ -, or  $\square$ -reductions so that the resulting cubic graph, which we denote by  $h(J_1)'$  is 2-connected. We see that  $h(J_1)'$  has 14 vertices, and thus according to Proposition 2.7 either  $\chi'(h(J_1)') = 3$  or  $h(J_1)'$  is a 3-chromatic expansion of  $P_{10}$ . Now Theorem 2.6 implies that either  $\chi'_h(J_1) = 3$  or  $h(J_1)$  is a 3-chromatic expansion of  $P_{10}$ . Since  $h(J \cup H_1)$  is a 3-chromatic expansion of  $h(J_1)$ , it follows that  $\chi'_h(J \cup H_1) = 3$  or  $h(J \cup H_1)$  is a 3-chromatic expansion of  $P_{10}$ . Since  $\chi'_h(H_i) = 3$  if  $u_i \neq v_i$ , it follows that  $\chi'_h(H) = 3$  or  $h(H)$  is a 3-chromatic expansion of  $P_{10}$  given by  $v \rightarrow A_v$ ,  $v \in V(P_{10})$  where we may assume that all cycles  $C'(v)$ ,  $v \in V(H_{\mathcal{C}}) \setminus X$  belong to some  $A_v$ , except for possibly  $C'(u_1)$ . Thus, Lemma 3.2 implies that  $h(H)$  is  $\mathcal{C}$ -compatible and this completes the proof of Case 2.1.

Case 2.2: Suppose each block of  $G_{\mathcal{C}}$  has at most 2 odd vertices.

If each block of  $G_{\mathcal{C}}$  has at most 2 odd vertices, then it is seen that  $G_{\mathcal{C}}$  contains 2 disjoint block chains  $(B_0)_{\mathcal{C}}$  and  $(B_1)_{\mathcal{C}}$  (not having any common blocks) where the endblocks of the block chains each contain exactly one odd vertex of  $G_{\mathcal{C}}$ . We may assume  $v_1, v_2$  and  $v_3, v_4$  belong to the endblocks of  $(B_0)_{\mathcal{C}}$  and  $(B_1)_{\mathcal{C}}$ , respectively. By Lemma 5.1, there exists subgraphs  $(H_0)_{\mathcal{C}} \subseteq (B_0)_{\mathcal{C}}$  and  $(H_1)_{\mathcal{C}} \subseteq (B_1)_{\mathcal{C}}$  where  $(H_0)_{\mathcal{C}}, (H_1)_{\mathcal{C}} \in \mathcal{F}_2^*$  and moreover,  $v_1, v_2$  and  $v_3, v_4$  are exactly the odd vertices of  $(H_0)_{\mathcal{C}}$  and  $(H_1)_{\mathcal{C}}$ , respectively. Let  $H_{\mathcal{C}} = (H_0)_{\mathcal{C}} \cup (H_1)_{\mathcal{C}}$ . The graph  $H_{\mathcal{C}}$  belongs to either  $\mathcal{F}_2^* \vee_0 \mathcal{F}_2^*$ ,  $\mathcal{F}_2^* \vee_1^e \mathcal{F}_2^*$ , or  $\mathcal{F}_2^* \vee_1^{oe} \mathcal{F}_2^*$ . If  $H_{\mathcal{C}} \in \mathcal{F}_2^* \vee_0 \mathcal{F}_2^*$ , then  $\chi'(h(H)) = 3$  (according to Proposition 5.2 (ii)) In this case  $h(H)$  is  $\mathcal{C}$ -compatible. Thus we may assume that either  $H_{\mathcal{C}} \in \mathcal{F}_2^* \vee_1^e \mathcal{F}_2^*$  or  $H_{\mathcal{C}} \in \mathcal{F}_2^* \vee_1^{oe} \mathcal{F}_2^*$ , and  $(H_0)_{\mathcal{C}}$  and  $(H_1)_{\mathcal{C}}$  intersect at a vertex  $u$ .

Let  $(H_u)_{\mathcal{C}}$  be the subgraph of  $H_{\mathcal{C}}$  which is the union of the blocks of  $H_{\mathcal{C}}$  which contain  $u$ . Let  $h(H_u)'$  be the graph obtained from  $h(H_u)$  by reducing all 2-, 3-, or 4-cycles  $C_{h(H_u)}(v) \subseteq h(H_u)$  via  $\circ$ -,  $\Delta$ -, or  $\square$ -reductions (where as usual, bridgelessness is preserved). The resulting graph has at most 16 vertices, and according to Proposition 2.7 either  $\chi'(h(H_u)') = 3$  or  $h(H_u)'$  is a 3-chromatic expansion of  $P_{10}$ . It then follows from Theorem 2.6 that either  $\chi'_h(H_u) = 3$  or  $h(H_u)$  is a 3-chromatic expansion of  $P_{10}$ . If  $\chi'_h(H_u) = 3$ , then by Lemma 4.2(b) we have  $\chi'_h(H) = 3$ . In this case,  $h(H)$  is  $\mathcal{C}$ -compatible. On the other hand, if  $h(H_u)$  is a 3-chromatic expansion of  $P_{10}$ , then  $h(H)$  is a 3-chromatic expansion of  $P_{10}$  where we may assume that the expansion has a representation  $v \rightarrow A_v$ ,  $v \in V(P_{10})$  such that all cycles  $C_{h(H)}(v)$ ,  $v \in V(H_{\mathcal{C}})$  belong to some  $A_v$  except for possibly  $C_{h(H)}(u)$ . Lemma 3.2 implies that  $h(H)$  is  $\mathcal{C}$ -compatible. This completes the proof of case 2.2.  $\square$

**Proof of Theorem 1.2.** We suppose again that  $G$  is a 2-connected, cubic graph and let  $(X, \mathcal{C})$  be a pseudo 2-factor of  $G$  where  $|X|$  plus the number of odd cycles in  $\mathcal{C}$  is at most 4. We may assume that the theorem holds for any graph with fewer vertices than  $G$ .

Fig. 11. A vine  $P_1, \dots, P_k$ .

Suppose that  $G$  has a non-trivial 3-edge cut. Then  $G$  can be expressed as a vertex expansion  $G = (G_1; u; e_1, e_2, e_3) \otimes (G_2; v; f_1, f_2, f_3)$  where  $v(G_i) < v(G)$ ,  $i = 1, 2$ . For  $i = 1, 2$  let  $(X_i, C_i)$  be the pseudo 2-factor of  $G_i$  obtained from  $G$  in the natural way. Then  $|X_i|$  plus the number of odd cycles in  $C_i$  is at most 4. Thus  $o(G_i) \leq o(G)$ , and hence by assumption,  $G_1$  and  $G_2$  each admit double cycle covers  $\mathcal{D}_1$  and  $\mathcal{D}_2$ , respectively. Now one can construct a cycle double cover  $\mathcal{D}$  via splicing  $\mathcal{D}_1$  and  $\mathcal{D}_2$  together.

If we now assume that  $G$  has no non-trivial 3-edge cuts, then Theorem 6.1 implies that  $G_C$  has a bridgeless degree compatible subgraph  $H_C$  for which  $h(H)$  is  $C$ -compatible. By Lemma 3.4, we can construct a cycle double cover for  $G$ . This completes the proof of the theorem.  $\square$

## 7. Vines

Let  $P$  be a path  $v_0 v_1 \dots v_n$  and let  $P_1, \dots, P_k$  be a collection of paths which intersect  $P$  at exactly their terminal vertices, where for each  $i$ ,  $P_i$  has terminal vertices  $v_{t(i)}$  and  $v_{h(i)}$  and  $t(i) < h(i)$ . If the paths  $P_i$ ,  $i = 1, \dots, k$  are internally vertex-disjoint and satisfy,

- (i)  $t(1) = 0$ ,  $h(k) = n$ .
- (ii)  $t(i) < t(i+1) \leq h(i) < h(i+1)$ ,  $i = 1, \dots, k-1$ .
- (iii)  $h(i) < t(i+2)$ ,  $i = 1, \dots, k-2$

then we say that  $P_1, \dots, P_k$  form a *vine* along  $P$ . Note that a vine may consist of just one path. We say that vertices  $u$  and  $v$  are *joined by a vine* if there exists a path  $P$  from  $u$  to  $v$  and a vine along  $P$  (see Fig. 11).

Let  $G$  be a graph and let  $H$  be a subgraph. Let  $P$  be a path from  $u$  to  $v$  in  $H$  and let  $P_1, \dots, P_k$  be a vine along  $P$  where each  $P_i$  intersects  $H$  only at its terminal vertices. Then we say that  $P_1, \dots, P_k$  is an  $H$ -vine. In this case, we say that there is an  $H$ -vine from  $u$  to  $v$  in  $H$ .

Let  $P_1, \dots, P_k$  be a vine along  $P = v_0 v_1 \dots v_n$  as above. We shall now define what we call the *circuit of the vine*  $C_{P_1, \dots, P_k}$ . If  $k = 1$ , let  $C_{P_1, \dots, P_k} = P \cup P_1$ . If  $k = 2l + 1$ ,  $l \geq 1$ , then let

$$\begin{aligned} C_{P_1, \dots, P_k} = & P_1 \cup P[v_{h(1)}, v_{t(3)}] \cup \dots \cup P_{2l-1} \cup P[v_{h(2l-1)}, v_{t(2l+1)}] \dots \\ & \cup P_{2l+1} \cup P[v_0, v_{t(2)}] \cup P_2 \cup \dots \cup P[v_{h(2i)}, v_{t(2i+2)}] \\ & \cup P_{2i+2} \cup \dots \cup P[v_{h(2l)}, v_n]. \end{aligned}$$

If  $k = 2l$ ,  $l \geq 1$ , then let

$$\begin{aligned} C_{P_1, \dots, P_k} = & P_1 \cup P[v_{h(1)}, v_{t(3)}] \cup \dots \cup P_{2l-1} \cup P[v_{h(2l-1)}, v_{t(2l+1)}] \dots \cup P_{2l-1} \\ & \cup P[v_{h(2l-1)}, v_{2l}] \cup P[v_0, v_{t(2)}] \cup P_2 \cup \dots \cup P[v_{h(2i)}, v_{t(2i+2)}] \\ & \cup P_{2i+2} \cup \dots \cup P[v_{h(2l-2)}, v_{2l}] \cup P_{2l}. \end{aligned}$$

Let  $G$  be a connected cubic graph containing a subgraph  $H$  which is homeomorphic to a cubic graph  $\tilde{H}$ . For each edge  $\tilde{e} \in E(\tilde{H})$ , let  $[\tilde{e}]_H$  be the corresponding path in  $H$ , and for any subgraph  $\tilde{I} \subseteq \tilde{H}$ , we let  $[I]_H$  be the corresponding subgraph in  $H$ . We leave the verification of the following theorem to the reader.

**Theorem 7.1.** *Suppose for any two edges  $\tilde{e}, \tilde{f} \in E(\tilde{H})$  it holds that if there is an  $H$ -vine from a vertex of  $[\tilde{e}]_H$  to a vertex of  $[\tilde{f}]_H$ , then  $\tilde{e}$  and  $\tilde{f}$  are incident in  $\tilde{H}$ . Then the graph  $G$  is an expansion of  $H$ .*

**Theorem 7.2.** *Let  $G$  be a connected, cubic graph and let  $H$  be a subgraph homeomorphic to  $\tilde{H} \simeq P_{10}$ . Suppose  $\tilde{e}, \tilde{f} \in E(\tilde{H})$  are two non-incident edges. If there is an  $H$ -vine from a vertex  $v_0 \in [\tilde{e}]_H$  to a vertex  $v_n \in [\tilde{f}]_H$ . Then for some such vine  $P_1, \dots, P_k$  it holds that  $\chi'_h(H \cup P_1 \cup \dots \cup P_k) = 3$ .*

**Proof.** Suppose that there are non-incident edges  $\tilde{e}, \tilde{f} \in E(\tilde{H})$  for which there is an  $H$ -vine from a vertex of  $[\tilde{e}]_H$  to a vertex of  $[\tilde{f}]_H$ . Pick such a vine having a fewest number of paths, say  $P_1, \dots, P_k$ , and assume that it is an  $H$ -vine along a path  $P \subseteq H$  from a vertex  $v_0 \in [\tilde{e}]_H$  to a vertex  $v_n \in [\tilde{f}]_H$ . Given that  $\tilde{H} \simeq P_{10}$ ,  $\tilde{H}$  has a 2-factor  $\tilde{C}_1$  and  $\tilde{C}_2$  being two 5-cycles where  $\tilde{e} \in E(\tilde{C}_1)$  and  $\tilde{f} \in \tilde{C}_2$ . If the vine consists of only one path  $P_1$ , then  $h(H \cup P_1)$  has a 2-factor consisting of two 6-cycles,  $[\tilde{C}_i]_H$ ,  $i = 1, 2$ . In this case,  $\chi'_h(H \cup P_1) = 3$ . We suppose therefore that the vine has more than one path. Since we chose  $P_1, \dots, P_k$  to have as few paths as possible, we have that the distance between  $\tilde{e}$  and  $\tilde{f}$  in  $\tilde{H}$  equals 2, and moreover, there is a path  $u\tilde{e}v\tilde{g}w\tilde{f}z$  in  $\tilde{H}$  such that  $P \subseteq [u\tilde{e}v\tilde{g}w\tilde{f}z]_H$ .

Let  $C_i = [\tilde{C}_i]_H$ ,  $i = 1, 2$ . Now  $(C_1 \cup C_2) \nabla C_{P_1 \dots P_k}$  is a cycle in  $G$ , which is also a hamilton cycle in  $h(H \cup P_1 \cup \dots \cup P_k)$ . thus we have that  $\chi'_h(H \cup P_1 \cup \dots \cup P_k) = 3$ .  $\square$

From the above, we obtain the following corollary.

**Corollary 7.3.** *Let  $G$  be a connected cubic graph and let  $H$  be a subgraph homeomorphic to  $P_{10}$ . Then either  $G$  is an expansion of  $P_{10}$ , or there is an  $H$ -vine  $P_1, \dots, P_k$  such that  $\chi'_h(H \cup P_1 \cup \dots \cup P_k) = 3$ .*

We also see that Theorem 2.5 is a consequence of the above result.

## Appendix A.

Let  $G$  be a 2-connected cubic graph with  $o(G) = 4$  and let  $(X, \mathcal{C})$  be a pseudo 2-factor of  $G$  where  $|X|$  plus the number of odd cycles in  $\mathcal{C}$  equals 4. Let  $G_{\mathcal{C}}$  be the graph obtained from  $G$  by contracting the cycles of  $\mathcal{C}$ .



**Theorem A.1.** *Suppose that the odd vertices of  $G_C$  are contained in a block  $B_C$ . Then one of the three statements holds:*

- (i)  $G_C$  contains a degree-compatible subgraph  $H_C$  satisfying one of (3.1.1)–(3.1.8) for which  $h(H)$  is  $\mathcal{C}$ -compatible.
- (ii)  $G_C$  contains a degree-compatible subgraph  $H_C$  where each odd degree vertex has degree three and for which  $h(H)$  is  $\mathcal{C}$ -compatible.
- (iii)  $G$  contains a non-trivial 3-edge cut which separates a vertex of  $X$  or an odd cycle of  $\mathcal{C}$  from the other vertices of  $X$  and odd cycles of  $\mathcal{C}$ .

**Proof.** The odd vertices of  $G_C$  are easily seen to be exactly the odd vertices of  $B_C$ . Since  $B_C$  is a block with more than one edge, Lemmas 3.5 and 3.6 imply that it contains a degree compatible subgraph  $H_C \in \mathcal{F}_4$  (which is also degree-compatible in  $G_C$ ) satisfying one of (3.1.1)–(3.1.9). If  $H_C$  satisfies one of (3.1.1)–(3.1.8), then following the proof of Theorem 6.1, case 1.1, the graph  $h(H)$  would be  $\mathcal{C}$ -compatible. In this case, (i) holds. So we may assume that  $H_C$  satisfies (3.1.9), and moreover,  $G_C$  contains no degree compatible subgraph in  $\mathcal{F}_4$  which satisfies one of (3.1.1)–(3.1.8). We shall assume that  $v_1, \dots, v_4$  are exactly the odd vertices of  $G_C$ . We shall let  $C'(v)$ ,  $h(H)'$ ,  $G'_C$ , and  $G'$  be as defined in the proof of Theorem 6.1.

We shall first show that  $h(H)$  is a 3-chromatic expansion of  $P_{10}$ . We have that  $v(h(H)') = 18$ . If  $\chi'(h(H)') = 3$ , then  $\chi'_h(H) = 3$  (by Lemma 1.7 and Corollary 2.2). It would then follow from Lemma 3.1 that  $h(H)$  is  $\mathcal{C}$ -compatible. This being the case, we may assume that  $\chi'(h(H)') = \chi'_h(H) = 4$ . According to [3], there are only 3 different cubic graphs of order 18 having girth at least 5 and chromatic index  $\chi' = 4$ . Two such graphs are obtained by performing a 4-edge expansion with the graphs  $P_8$  and  $P_{10}$ . The third graph is obtained by performing a vertex expansion at one vertex  $u$  of  $P_{10}$ , where  $u \rightarrow P_{10}$ .

Since  $H_C \in \mathcal{F}_4$ , we have that  $H_C = H'_C \cup F'_C$  where  $H'_C$  is homeomorphic to one of the graphs in Fig. 9 and  $F'_C$  is homeomorphic to a forest in Fig. 10. From the table in Section 3, we see that there is only one possibility for  $H'_C$  and  $F'_C$ ; the graph  $F'_C$  is homeomorphic to  $F_2$ , and  $H'_C$  is homeomorphic to  $H_C^6$ . Given that we are assuming that  $G_C$  has no degree-compatible subgraphs in  $\mathcal{F}_4$  satisfying one of (3.1.1)–(3.1.8), we have that the vertices  $v_{13}$  and  $v_{24}$  (as specified by Lemma 3.5) form a 2-separating set for  $G_C$  which separates each pair of vertices  $v_i$  and  $v_j$ ,  $i \neq j$ . Let  $u_1 = v_{13}$  and  $u_2 = v_{24}$ . The graph  $H'_C$  consists of 4 internally vertex-disjoint paths  $P_C^1, P_C^2, P_C^3, P_C^4$  between  $u_1$  and  $u_2$ , where  $v_i \in V(P_C^i)$ ,  $i = 1, \dots, 4$ . The graph  $F'_C$  is homeomorphic to  $F_2$  and consists of 4 internally vertex-disjoint paths originating at  $u_1$  and terminating at  $v_i$ . One of these paths contains  $u_2$ , and we may assume that this path terminates at  $v_4$ . For  $i = 1, 2, 3$ , we denote the path terminating at  $v_i$  by  $Q_C^i$ , and we denote the path terminating at  $v_4$  by  $P_C^5 \cup Q_C^4$  where  $P_C^5$  is the portion of the path between  $u_1$  and  $u_2$ , and  $Q_C^4$  is the portion of the path between  $u_2$  and  $v_4$  (see Fig. 12).

The cycles  $C'(u_1)$  and  $C'(u_2)$  are vertex-disjoint cycles of  $h(B)'$  having lengths 8 and 6, respectively. Suppose  $h(H)'$  is a 4-edge expansion of  $P_8$  with  $P_{10}$ . Let  $A = \{f_1, f_2, f_3, f_4\}$  be the 4-edge cut formed via this expansion. Then  $h(B) \setminus \{f_1, f_2, f_3, f_4\}$  has exactly 2 components  $K_1$  and  $K_2$  having 10 and 8 vertices, respectively. Suppose first that neither



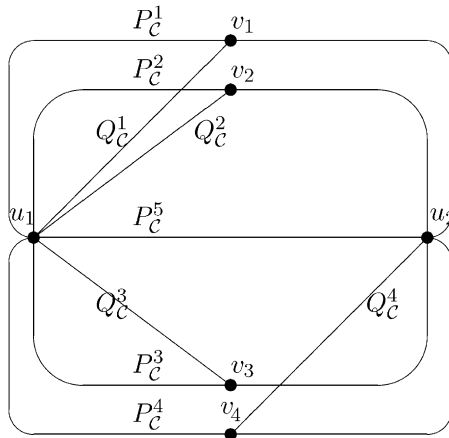


Fig. 12. The graph  $H_C$ .

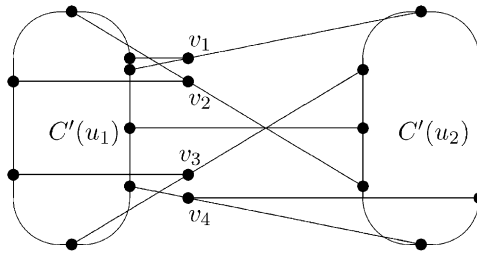


Fig. 13. The graph  $h(H)'$ .

$C'(u_1)$  nor  $C'(u_2)$  contain edges of  $A$ . Then either both cycles belong to one component, or they belong to separate components. The former is impossible considering that each component has at most 10 vertices. The latter is also seen to be impossible since there are 5 edge-disjoint paths between  $u_1$  and  $u_2$  in  $H_C$ , and hence no 4-edge cut in  $h(H)'$  can separate  $C'(u_1)$  and  $C'(u_2)$ . We conclude that at least one of the cycles contains edges of  $A$  (see Fig. 13).

Suppose  $C'(u_1)$  contains no edges of  $A$ , but  $C'(u_2)$  does. Then  $C'(u_1) \subseteq K_1$ ; for otherwise, if  $C'(u_1) \subseteq K_2$ , then it would follow that  $C'(u_2) \subseteq K_1$ . Given that  $C'(u_2)$  contains at least 2 edges of  $A$ , one sees upon examination of  $H_C$  that  $K_1$  would contain at least 3 of the vertices  $v_i$ ,  $i = 1, \dots, 4$  and hence  $v(K_1) \geq 8 + 3 = 11$  vertices. This yields a contradiction. We may therefore assume that  $C'(u_1)$  contains edges of  $A$ , and hence it must have at least 2 such edges.

Suppose  $C'(u_2)$  contains no edges of  $A$ . Given  $C'(u_1)$  contains at least 2 edges of  $A$ , one sees by inspecting  $H_C$  that for at least 2 of the vertices  $v_i$ ,  $i = 1, \dots, 4$ , no edge of  $A$  is incident with  $v_i$ . Thus the component( $K_1$  or  $K_2$ ) containing  $C'(u_2)$  would have at least  $6 + 5 = 11$  vertices. This yields a contradiction. We may therefore assume that  $C'(u_1)$  and  $C'(u_2)$  both contain edges of  $A$ , and hence they contain 2 edges apiece.

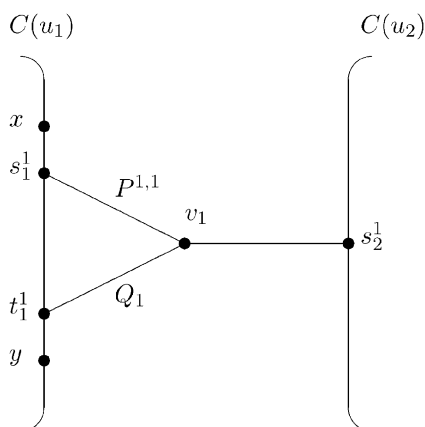


Fig. 14.

We now have that no edge of  $A$  is incident with the vertices  $v_i$ ,  $i = 1, \dots, 4$  and hence the neighbours of  $v_i$  in  $h(H)$  belong to the same component ( $K_1$  or  $K_2$ ) as  $v_i$ . Thus, neither  $K_1$  nor  $K_2$  can contain 3 or more of the vertices  $v_i$ , and each component contains 2 vertices apiece. Suppose  $v_i$  and  $v_j$  belong to  $K_2$ . Given that  $K_2$  is in the  $P_8$  part of the 4-edge expansion and no edge of  $A$  is incident with  $v_i$  or  $v_j$ , it follows that  $\text{dist}_{K_2}(v_i, v_j) \leq 2$ . However, upon inspection of  $H_C$ , one sees that for  $i \neq j$ ,  $\text{dist}_{h(H)'}(v_i, v_j) \geq 3$ . Here we reach a final contradiction. We conclude that  $h(H)'$  cannot be a 4-edge expansion of  $P_8$  with  $P_{10}$ . Similar arguments also demonstrate that  $h(H)'$  is not a vertex-expansion of  $P_{10}$  where for a vertex  $u \in V(P_{10})$  we expand the vertex via  $u \rightarrow P_{10}$ .

From the above, Proposition 2.7 implies that  $h(H)'$  must be a 3-chromatic expansion of  $P_{10}$ . Hence  $h(H)$  is also a 3-chromatic expansion of  $P_{10}$ , and we let  $v \rightarrow A_v$ ,  $v \in V(P_{10})$  be a representation of this expansion. For each  $v$ , let  $A'_v$  be the subgraph of  $h(H)$  induced by those edges of  $h(H)$  coinciding with those in  $A_v$ . If one of the cycles  $C(u_i)$ ,  $i = 1, 2$  belongs to some  $A'_v$ , then all but one of the cycles  $C(v)$ ,  $v \in V(B_C) \setminus X$  belong to  $A'_v$ 's, and as was demonstrated before,  $h(H)$  is  $\mathcal{C}$ -compatible in this case. Thus, we may assume that neither  $C(u_1)$  nor  $C(u_2)$  are contained in any  $A'_v$ . Thus each cycle intersects exactly 5 of the subgraphs  $A'_v$ .

Suppose  $P_C$  is a path in  $G_C$  and  $u$  is one of its endvertices. We define a *stem-vertex* of  $P$  in the following way: if  $u \in X$ , then it is a stem vertex. Otherwise, we define a vertex of  $C(u)$  to be a stem-vertex if it is a separating vertex of  $P$ .

For  $i = 1, 2$  and  $j = 1, \dots, 5$  let  $s_i^j$  denote the stem-vertex of  $P^j$  on  $C(u_i)$ . For  $j = 1, 2, 3$  let  $t_1^j$  denote the stem-vertex of  $Q^j$  on  $C(u_1)$  and let  $t_2^4$  denote the stem-vertex of  $Q^4$  lying on  $C(u_2)$ . Let  $P^{1,1}$  denote the portion of  $P^1$  lying between  $C(v_1)$  and  $v_1$ . Let  $x$  and  $y$  denote the stem-vertices lying on either side of  $s_1^1$  and  $t_1^1$ , and let  $C_1[x, y]$  denote the portion of  $C_1$  between  $x$  and  $y$  which contains  $s_1^1$  and  $t_1^1$  (see Fig. 14). Let  $J = C_1[s_1^1, t_1^1] \cup P^{1,1} \cup Q^1$ . One can show that  $\{x, y, s_2^1\}$  is a 3-separating set of  $H$  which separates  $H$  into two subgraphs  $H_1$  and  $H_2$  (so that  $H_1 \cap H_2 = \{x, y, s_2^1\}$ ) where  $C_1[x, y] \cup P^1 \cup Q^1 \subseteq H_1$ . If there is a  $H$ -vine in  $G$  from a vertex in  $J$  to a vertex in  $H_2$ , then we could modify  $H_C$  to obtain

a degree-compatible graph of  $G_C$  for which  $h(H)$  is 3-edge colourable and hence  $\mathcal{C}$ -compatible (we leave the verification of this to the reader). In addition, each odd degree vertex in such a subgraph has degree three, in which case (ii) holds. We may thus assume that no such  $H$ -vine exists. Let  $x'$  be the vertex of  $C_1[x, y]$  closest to  $x$ , where  $x'$  is joined to a vertex in  $J$  by an  $H$ -vine in  $G$ . We define  $y'$  analogously for  $y$ . Let  $s_2^{1'}$  be the vertex of  $P^1$  closest to  $s_2^1$  which is joined by an  $H$ -vine to a vertex in  $J$ . Now  $\{x', y', s_1^{1'}\}$  is a 3-separating set in  $G$ . Consequently,  $G$  contains a non-trivial 3-edge cut which separates the odd cycle or vertex in  $G$  corresponding to  $v_1$  from the odd cycles or vertices in  $G$  corresponding to  $v_2, v_3$ , and  $v_4$ . In this case, (iii) is seen to hold. This concludes the proof of the theorem.  $\square$

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